Conformally Kähler, Einstein-Maxwell metrics (after Isaque Viza de Souza)

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Let (M, J) be a compact Kähler manifold.

Definition: A Hermitian metric $\tilde{g}$ on (M, J) is a conformally Kähler, Einstein-Maxwell metric (cKEM metric for short) if it satisfies the following three conditions:

(a) The scalar curvature $s_{\tilde{g}}$ of $\tilde{g}$ is constant.

(b) There exists a positive smooth function f on M such that $g = f^2 \tilde{g}$ is Kähler.

(c) The Hamiltonian vector field $K = J\mathrm{grad}_g f$ for f is a Killing vector field for both g and $\tilde{g}$.
(Necessarily, $K - iJK$ is a holomorphic vector field.)

This may be considered as a "conformally Kähler Yamabe problem".

If $K = 0$, i.e. f is a constant function then g is a Kähler metric of constant scalar curvature. (Yau-Tian-Donaldson conjecture)

We start with a compact Kähler manifold M with fixed Kähler class and fixed Killing vector field K in the Lie algebra of the maximal torus of the automorphism group, and
search for g such that $f_K^{-2}g$ has constant scalar curvature, $\mathrm{grad}_g f_K = K$.
"Conformally Einstein-Maxwell Kähler metric problem" :
View point taken by Apostolov-Maschler.
(Extended Yau-Tian-Donaldson conjecture)

Examples of the conformally Kähler, Einstein metrics :

Page metric on the one-point-blow-up of $\mathbf{CP}^2$ (1978),

Chen-LeBrun-Weber on the two-point-blow-up of $\mathbf{CP}^2$ (2008).

Apostolov-Calderbank-Gauduchon on 4-orbifolds (2015),

Bérard-Bergery on $\mathbf{P}^1$-bundles over Fano Kähler-Einstein manifolds (1982).

Examples of non-Einstein cKEM examples :

LeBrun's ambitoric examples on $\mathbf{CP}^1 \times \mathbf{CP}^1$, the one-point-blow-up of $\mathbf{CP}^2$, and Hirzebruch surfaces (2016),

Koca-Tϕnnesen-Friedman on ruled surfaces of higher genus (2016).

Apostolov and Maschler initiated a study in the **framework similar to the Kähler geometry**,

and set the existence problem of cKEM metrics in the **Donaldson-Fujiki picture**.

In particular, fixing a Kähler class, they defined an **obstruction to the existence of cKEM metrics** in a similar manner to the Kähler-Einstein and cscK cases.

They further studied the toric surfaces and showed the equivalence between the existence of cKEM metrics and toric K-stability on toric surfaces with convex quadrilateral moment map images, extending earlier works by Legendre and Donaldson.

Futaki-Ono (JMSJ 2018, Outstanding Paper Prize) studied
for which Killing vector field K we can find a cKEM metric.

We showed that, fixing a Kähler class, such Killing vector fields are critical points of certain **volume functional**.

We also showed that, for toric manifolds, this idea gives an efficient way to decide **which vector fields in the Lie algebra of the torus can have a solution of the cKEM problem**.

The idea is similar to the cases of **Kähler-Ricci solitons** and **Sasaki-Einstein metrics**.

A **Kähler-Ricci soliton** is a Kähler metric with its Kähler form $\omega \in c_1(M)$

such that there exists a Killing Hamiltonian vector field
$X \in \mathfrak{h}$,
the Lie algebra of the maximal torus of the automorphism group,
satisfying

$$\begin{aligned} \mathrm{Ric}_\omega &= \omega + L_{JX}\omega \\ &= \omega + i\partial\overline{\partial} f_X. \end{aligned}$$

Find out, <u>for which X</u>, there is a Kähler form ω satisfying the Kähler-Ricci soliton equation.

Let g be an arbitrary Kähler metric with its Kähler class $\omega_g \in c_1(M)$, and let h_g be a smooth function such that

$$\mathrm{Ric}_g - \omega_g = i\partial\overline{\partial} h_g.$$

Tian and Zhu (2002) defined a functional $\mathrm{Fut}_X : \mathfrak{g} \to \mathbf{R}$ by

$$\mathrm{Fut}_X(Y) = \int_M (JY)(h_g - f_X) e^{f_X} \omega_g^m$$

where f_X is the Hamiltonian function of X with the normalization

$$\int_M e^{f_X} \omega_g^m = \int_M \omega_g^m.$$

Fut_X is independent of the choice of $\omega_g \in c_1(M)$, and if there exists a Kähler-Ricci soliton for X then Fut_X vanishes identically.

To find such X with vanishing F_X, they considered the **weighted volume functional** $V : \mathfrak{g} \to \mathbf{R}$ defined by

$$V(Z) = \int_M e^{u_Z} \omega_g^m$$

where u_Z is the Hamiltonian function of $Z \in \mathfrak{g}$ with the the normalization

$$\int_M u_Z e^{h_g} \omega_g^m = 0.$$

Tian and Zhu showed that

- $V(X)$ is independent of ω_g,

- $dV_X(Y) = c\,\mathrm{Fut}_X(Y)$ with a constant c,

- V is a strictly convex proper function,

- there is a unique minimum X.

This minimum X is the right choice to solve the Kähler-Ricci soliton equation.

A similar story holds for the Sasaki-Einstein metrics. Martell-Sparks-Yau (2008).

Let S be a toric Sasaki manifold, i.e. its Kähler cone $C(S)$ is toric.

The deformation space of the toric Sasaki structure is described by a subspace H in the Lie algebra $\mathfrak{g}$ of the torus consisting of Reeb vector fields. That is, a deformation of Sasaki structure corresponds to a deformation of the Reeb vector field.

Let $V : H \to \mathbf{R}$ be the volume functional of the space of Sasaki manifolds.

The derivative of $dV_\xi : \mathfrak{g}_0 (= T_\xi H) \to \mathbf{R}$ is equal to the natural obstruction for the existence of Sasaki-Einstein metric.

Since V is strictly convex and proper on H, there is a unique minimum. This is the right choice of the Reeb vector field to look for a Sasaki-Einstein metric.

Let us turn to our cKEM problem.

Let G be a maximal torus of a maximal reductive subgroup of the automorphism group,

and take $K \in \mathfrak{g} := \mathrm{Lie}(G)$.

Let ω_0 be a Kähler form, and $\Omega = [\omega_0] \in H^2_{\mathrm{DR}}(M, \mathbf{R})$ be a fixed Kähler class.

We wish to find a G-invariant Kähler metric g with its Kähler form $\omega_g \in \Omega$ such that

(i) $\tilde{g} = f^{-2} g$ is a cKEM metric,

(ii) $J\mathrm{grad}_g f = K$.

But we need to know whether we have chosen the right $K \in \mathfrak{g}$.

If we have chosen a wrong one we will never get to a cKEM metric.

The right one has to have vanishing obstruction.

Denote by $\mathcal{K}_\Omega^G$ the space of G-invariant Kähler metrics g with $\omega_g \in \Omega$.

For any $(K, a, g) \in \mathfrak{g} \times \mathbf{R} \times \mathcal{K}_\Omega^G$, there exists a unique function $f_{K,a,g} \in C^\infty(M, \mathbf{R})$ satisfying the following two conditions:

$$\iota_K \omega_g = df_{K,a,g}, \quad \int_M f_{K,a,g} \frac{\omega_g^m}{m!} = a.$$

Noting min$\{f_{K,a,g}(x) \,|\, x \in M\}$ is independent of g with $\omega_g \in \Omega$, we put

$$\begin{aligned} \mathcal{P}^G_\Omega &:= \{(K,a) \in \mathfrak{g} \times \mathbf{R} \,|\, f_{K,a,g} > 0\}, \\ \mathcal{H}^G_\Omega &:= \left\{ \tilde{g}_{K,a} = f^{-2}_{K,a,g} g \,\middle|\, (K,a) \in \mathcal{P}^G_\Omega, g \in \mathcal{K}^G_\Omega \right\}. \end{aligned}$$

Fixing $(K,a) \in P^G_\Omega$, put

$$\mathcal{H}^G_{\Omega,K,a} := \{\tilde{g}_{K,a} \,|\, g \in \mathcal{K}^G_\Omega\}.$$

Hereafter the Kähler metric g and its Kähler form ω_g are often identified, and ω_g is often denoted by ω. Put

$$c_{\Omega,K,a} := \frac{\displaystyle\int_M s_{\tilde{g}_{K,a}} \ f_{K,a,g}^{-2m-1} \frac{\omega^m}{m!}}{\displaystyle\int_M \ f_{K,a,g}^{-2m-1} \frac{\omega^m}{m!}},$$

$$d_{\Omega,K,a} := \frac{\displaystyle\int_M s_{\tilde{g}_{K,a}} \ f_{K,a,g}^{-2m} \frac{\omega^m}{m!}}{\displaystyle\int_M \ f_{K,a,g}^{-2m} \frac{\omega^m}{m!}}.$$

Then $d_{\Omega,K,a}$, $c_{\Omega,K,a}$ are constants independent of the choice of $g \in \mathcal{K}_\Omega^G$ as shown by Apostolov-Maschler.

Then

$$F^G_{\Omega,K,a} : \mathfrak{g} \to \mathbf{R}$$

defined by

$$F^G_{\Omega,K,a}(H) := \int_M \left(\frac{s_{\tilde{g}_{K,a}} - c_{\Omega,K,a}}{f^{2m+1}_{K,a,g}} \right) f_{H,b,g} \frac{\omega^m}{m!}$$

is a linear function independent of the choice of $(g, b) \in \mathcal{K}^G_\Omega \times \mathbf{R}$.
(Apostolov-Maschler)

If there exists a constant scalar curvature metric in $\mathcal{H}_{\Omega,K,a}$, then $F^G_{\Omega,K,a}$ is identically zero.

Let us put further

$$\tilde{\mathcal{P}}_{\Omega}^{G} := \left\{ (K, a) \in \mathcal{P}_{\Omega}^{G} \,\middle|\, d_{\Omega,K,a} = 1 \right\}.$$

Consider the volume functional

$$\mathrm{Vol} : \tilde{\mathcal{P}}_{\Omega}^{G} \to \mathbf{R}$$

given by

$$\mathrm{Vol}(K, a) := \mathrm{Vol}(\tilde{g}_{K,a})$$

for $(K, a) \in \tilde{\mathcal{P}}_{\Omega}^{G}$.

The main result is the following volume minimization property of cKEM metrics.

Main Theorem:

Let $(K, a) \in \tilde{\mathcal{P}}^G_\Omega := \left\{ (K, a) \in \mathcal{P}^G_\Omega \,\middle|\, d_{\Omega,K,a} = 1 \right\}$.

Then if there exists a conformally Kähler, Einstein-Maxwell metric $\tilde{g}_{K,a} \in \mathcal{H}^G_{\Omega,K,a}$

then (K, a) is a critical point of

$$\mathrm{Vol} : \tilde{\mathcal{P}}^G_\Omega \to \mathbf{R}.$$

In fact,

$$d\,\mathrm{Vol}_{(K,a)} = \mathrm{const}\ F^G_{\Omega,K,a}.$$

A merit of the Main Theorem is to give a systematic computation of $F^G_{\Omega,K,a}$.

We use Maxima to compute $F^G_{\Omega,K,a}$ of $\mathbf{CP}^1 \times \mathbf{CP}^1$, the blow-up of $\mathbf{CP}^2$ at one point and other Hirzebruch surfaces.

The case of the one point blow up of $\mathbf{CP}^2$ Let Δ_p be the convex hull of $(0,0),(p,0),(p,1-p),(0,1),\ (0<p<1)$.

An affine linear function $f = a\mu_1 + b\mu_2 + c$ is positive on Δ_p if and only if

$$c, b+c, (1-p)b+pa+c, pa+c > 0.$$

(1) $a = \frac{p+2\sqrt{1-p}-2}{2p^2}, b = 0, 0 < p < 1$. $U(2)$-symmetry, LeBrun case.

(2) $a = -\frac{\sqrt{9p^2-8p}+p}{4p^2}, b = 0$. $\frac{8}{9} < p < 1$, $U(2)$-symmetry, LeBrun case.

(3) $a = \frac{\sqrt{9p^2-8p}-p}{4p^2}, b = 0$. $\frac{8}{9} < p < 1$, $U(2)$-symmetry, LeBrun case.

(4) $a = -\frac{\sqrt{p^4-4p^3+16p^2-16p+4}-p^2+4p-2}{2p^3-4p^2+12p-8}, b = -\frac{\sqrt{p^4-4p^3+16p^2-16p+4}}{p^3-2p^2+6p-4}$.
$0 < p < \alpha$, $U(1) \times U(1)$-symmtery. Existence unknow.

(5) $a = \frac{\sqrt{p^4-4p^3+16p^2-16p+4}+p^2-4p+2}{2p^3-4p^2+12p-8}, b = \frac{\sqrt{p^4-4p^3+16p^2-16p+4}}{p^3-2p^2+6p-4}$.
$0 < p < \alpha$, $U(1) \times U(1)$-symmtery. Existence unknow.

Here, α is the smallest positive root of $p^4 - 4p^3 + 16p^2 - 16p + 4 = 0$.

(6) $a = \frac{2\sqrt{-9b^2p^3+\left(21b^2+1\right)p^2+\left(1-16b^2\right)p+4b^2-1}+3bp^2+(1-2b)p}{6p^2-4p}.$

$U(1) \times U(1)$-symmtery. Existence unknown.

(7) $a = -\frac{2\sqrt{-9b^2p^3+\left(21b^2+1\right)p^2+\left(1-16b^2\right)p+4b^2-1}-3bp^2+(2b-1)p}{6p^2-4p}.$

$U(1) \times U(1)$-symmtery. Existence unknown.

For (4) and (5), numerical proof of existence has been given by Futaki-Ono arXiv:1803.06801 by checking toric K-stability.

Theorem (Viza de Souza, Ann. Global. Anal. Geom. 2021)

(A) There are solutions for (4) and (5).

(B) There are no solutions for (6) and (7).

Definition: Let Δ be a polytope in $\mathbb{R}^m$ containing the origin and f a positive affine linear function on Δ. The f-**twist** $\widetilde{\Delta} := T(\Delta)$ of Δ is the image of $T : \mathbb{R}^m \to \mathbb{R}^m$ where

$$T(x) = \tilde{x} = \frac{x}{f(x)}.$$

Lemma : If $\phi(x)$ is an affine linear function of x, then $\tilde{\phi}(\tilde{x}) := \frac{\phi(x)}{f(x)}$ is an affine function of $\tilde{x}$.

If $(\Delta, \mathbf{L} = \{L_1, \cdots, L_d\})$ is a labelled polytope, i.e.

$$\Delta = \{x \in \mathfrak{t}^* \mid L_i(x) \geq 0, \ i = 1, \cdots, d\},$$

then $(\widetilde{\Delta}, \widetilde{\mathbf{L}} = \{\widetilde{L}_1, \cdots, \widetilde{L}_d\})$, $\widetilde{L}_i(\tilde{x}) = L_i(x)/f(x)$ is a labelled polytope.

Put $\tilde{f}(\tilde{x}) := 1/f(x)$. Then $\tilde{f}$-twist of $\widetilde{\Delta}$ recovers Δ.

Lemma : If u is a symplectic potential of $(\Delta, \mathbf{L})$, then $\tilde{u}$ is a symplectic potential of $(\widetilde{\Delta}, \tilde{\mathbf{L}})$.

Theorem (Apostolov-Calderbank, arXiv:1810) : (Δ, u) corresponds to an extramal Kähler metric if and only if $(\widetilde{\Delta}, \tilde{u})$ corresponds to an $(f, m+2)$-extremal metirc for $g_{\tilde{u}}$, where the (f, w)-scalar curvature for a Kähler metric g is

$$Scal_{(f,w)}(g) = f^2 Scal(g) - 2(w-1) f \Delta_g f - w(w-1)|df|_g^2,$$

and $(f, m+2)$-scalar curvature for $g_{\tilde{u}}$ is a Killing potential.

Fact ; For $m = 2$, the scalar curvature of $f^{-2}g$ is equal to $Scal_{(f,4)}(g)$.

Corollary : For $m = 2$, if (Δ, u) corresponds to an extramal Kähler metric then $(\widetilde{\Delta}, \tilde{u})$ corresponds to a conformally Kähler, Einstein-Maxwell metric provided the Futaki invariant vanishes.

Remark : This corollary has been observed by Apostolov-Maschler.
Remrak : The above Theorem of Apostolov-Calderbank is more general, and expressed in terms of transverse Kähler structures on Sasakian manifolds.

Thus we only need to check the $\tilde{f}$-twist of cKEM metric is extremal or not.

Theorem (Legendre, 2011) : A quadrilateral in $\mathbb{R}^2$ corresponds to an extremal metric if and only if it is equipoised, i.e. if the vertices of the quadrilateral is v_1, v_2, v_3, v_4 in the natural order, then the L^2-projection ζ of the scalar curvature to the space of Killing potentials satisfy

$$\zeta(v_1) + \zeta(v_3) = \zeta(v_2) + \zeta(v_4).$$

Proof of Viza de Souza : The $\tilde{f}$-twist of (4) (resp. (5)) is equipoised. Thus it is extremal Kähler. Hence (4) (resp. (5)) is cKEM. In the case of (6) and (7) f can not be positive on the quadrilateral. Thus there is no solution.

Let $\mu : M \to \mathfrak{t}^*$ be the moment map.

Def (Lahdili) : For positive $v,\ w \in C^\infty(\Delta)$, we define the (v, w)-**scalar curvature** by

$$Scal_{(v,w)} = w^{-1}\left(v(\mu)Scal + 2\Delta v(\mu) + \sum_{i,j} v_{ij}(\mu)\right)$$

where v_{ij} denotes the Hessian with respect to an orthonormal basis of $\mathfrak{t}^*$.

Example : When $v(\mu) = f^{-2m+1}$ and $w(\mu) = f^{-2m-1}$, a constant (v,w)-scalar curvature metric is a conformally Kähler, Einstein-Maxwell metric.

Example (Eiji Inoue) : When M is Fano, $v = e^{<\xi,p>}$ for $\xi \in \mathfrak{t}$, $w = v(<\xi, p> + c)$, then a constant (v,w)-scalar curvature metric a gradient Kähler-Ricci soliton, where $J\xi$ is the soliton vector field.

$\Longrightarrow$ Fujiki-Donaldson picture for Kähler-Ricci solitons

$\Longrightarrow$ moduli theory for Kähler-Ricci solitons