# A conical approximation of constant scalar curvature Kähler metrics of Poincaré type and log K-semistability

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# Outline


1. Background (Kähler-Einstein case)
2. Conical approximation (Differential Geometry)
3. log K-stability (Algebraic Geometry)

# 1. Background (Kähler-Einstein case)

# Kähler metric and curvature

$X$ : $n$-dimensional complex manifold
$(z^1, z^2, ..., z^n)$ : local holomorphic coordinates

## Definition (Kähler metric)

A **Kähler metric** $\omega$ is a closed positive (1,1) form on $X$.

$$\omega = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz^i \wedge d\overline{z^j}, \ \ (g_{i\bar{j}})_{i,j} > 0.$$

## Definition (Ricci form and scalar curvature)

The **Ricci form** of $\omega$ is defined by

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial} \log \det(g_{i\bar{j}})_{i,j} \in c_1(X) = c_1(-K_X).$$

The **scalar curvature** of $\omega$ is defined by

$$S(\omega) = \mathrm{tr}_\omega \mathrm{Ric}(\omega) \in C^\infty(X, \mathbb{R}).$$

# cscK metrics

**Definition**

$\omega$ is a **constant scalar curvature Kähler (cscK) metric** if its scalar curvature is constant, i.e.,

$$S(\omega) = \text{const.}$$

**ex.**

- constant curvature metric on Riemann surface
- Kähler-Einstein metric

$$\exists \lambda \in \mathbb{R} \ \text{ s.t. } \ \mathrm{Ric}(\omega) = \lambda \omega$$

# Main problem

Let $(X, L_X)$ be a pair of a compact Kähler manifold $X$ and an ample line bundle $L_X$. Let $D$ be a smooth divisor.

**Problem**

*Does there exists a cscK metric on $X \setminus D$ with some singularities? Namely, solve the forth order nonlinear PDE on $X \setminus D$ :*

$$S(\omega + \sqrt{-1}\partial\overline{\partial}\phi) = \text{const}, \quad \phi \in C^{4,\alpha}(X \setminus D).$$

On local holomorphic coordinates,

$$\begin{cases} S(\omega + \sqrt{-1}\partial\overline{\partial}\phi) = -g_{\phi}^{i\bar{j}}\partial_i\partial_{\bar{j}}\log\det(g_{k\bar{l}} + \phi_{k\bar{l}}) = \text{const}, \\ \omega + \sqrt{-1}\partial\overline{\partial}\phi \quad = \quad \sqrt{-1}(g_{i\bar{j}} + \phi_{i\bar{j}})dz^i \wedge d\overline{z}^j > 0. \end{cases}$$

# Kähler-Einstein metrics with negative Ricci curvature

**Theorem (Kobayashi '84, Tian-Yau '87)**

*If* $K_X + D$ *is ample, there exists a unique Kähler-Einstein metric* $\omega \in c_1(K_X + D)$ *with Poincaré type singularities along* $D$ *such that*

$$\mathrm{Ric}\omega = -\omega \quad \text{on } X \setminus D.$$

**Theorem (Jeffres-Mazzeo-Rubinstein'11, Campana-Guenancia-Păun'13)**

*Take* $\beta_0 > 0$ *so that* $K_X + (1-\beta)D$ *is ample for all* $\beta \in (0, \beta_0)$*. There exists a unique Kähler-Einstein metric* $\omega_\beta \in c_1(K_X) + (1-\beta)c_1(D)$ *with cone singularities along* $D$ *for angle* $2\pi\beta$ *such that*

$$\mathrm{Ric}\omega_\beta = -\omega_\beta \quad \text{on } X \setminus D.$$

**Proof.**

Solve complex Monge-Ampère equations. □

# Conical approximation of Kähler-Einstein metrics

**Theorem (Guenancia '20)**

*Assume that $K_X + D$ is ample. Then, there is a family of Kähler-Einstein metrics with cone singularities of angle $2\pi\beta$ converges to a Kähler-Einstein metric of Poincaré type as $\beta \to 0$ in the sense of pointed Gromov-Hausdorff topology.*

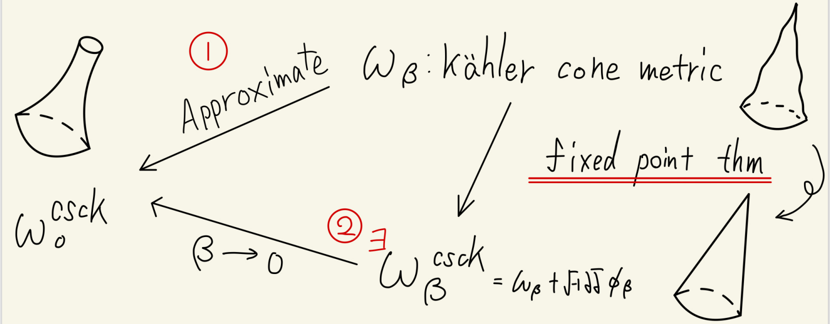
In this talk, we consider the analogue of Guenancia's result for cscK metrics.

## 2. Conical approximation (Differential Geometry)

# Main result and strategy

**Theorem (A, '22)**

*Assume that* $H^0(D, TD) = 0$ *and* $\mathrm{Aut}_0((X, L_X); D)$ *is trivial. If* $X \setminus D$ *has a cscK metric* $\omega_0^{cscK} \in c_1(L_X)$ *of Poincaré type, then* $X \setminus D$ *admits a cscK cone metric* $\omega_\beta^{cscK} \in c_1(L_X)$ *for sufficiently small angle* $2\pi\beta$*. Moreover,* $\omega_\beta^{cscK} \to \omega_0^{cscK}$ *as* $\beta \to 0$ *in the weighted Hölder space* $C_\eta^{4,\alpha}(X \setminus D)$ *for some* $-1 \ll \eta < 0$*.*



# Preliminaries

- $(X, L_X)$ : a polarized manifold
- $D \in |L_X|$ : a smooth hypersurface
- $\sigma_D \in H^0(X, L_X)$ : a defining section of $D$
- $H^0(D, TD) = 0$ and $\mathrm{Aut}_0((X, L_X); D)$ is trivial.
- $h_X$ : a Hermitian metric on $L_X$ with positive curvature
- $t := \log \|\sigma_D\|_{h_X}^{-2} \in \mathrm{PSH}(X \setminus D)$ ($t \to +\infty$ near $D$.)
- $\theta_X := \sqrt{-1}\partial\overline{\partial}t$ : a Kähler metric on $X$
- $\theta_D := \theta_X|_D$

# Kähler metrics of Poincaré type

## Definition (Poincaré type Kähler metrics)

We say that $\omega = \theta_X + \sqrt{-1}\partial\overline{\partial}s$ is a **Kähler metric of Poincaré type in the class** $[\theta_X]$ iff it is quasi-isometric to the model cusp metric

$$\frac{\sqrt{-1}dz^1 \wedge d\overline{z}^1}{|z^1|^2 \log^2 |z^1|^2} + \sum_j \sqrt{-1}dz^j \wedge d\overline{z}^j$$

and $s = O(\log\log|z^1|^{-2})$ near $D = \{z^1 = 0\}$.

## Definition (the average of scalar curvature)

$$\underline{S} := \frac{\int_{X\setminus D} S(\omega)\omega^n}{\int_{X\setminus D}\omega^n} = \frac{-n(K_X + L_X)L_X^{n-1}}{L_X^n}$$

$$\underline{S}_D := \frac{\int_D S(\theta_D)\theta_D^{n-1}}{\int_D \theta_D^{n-1}} = \frac{-(n-1)(K_X + L_X)|_D(L_X|_D)^{n-2}}{(L_X|_D)^{n-1}}$$

# Auvray's work

**Theorem (Auvray '13)**

*If $X \setminus D$ have a cscK metric of Poincaré type, then the following inequality holds :*

$$\underline{S} < \underline{S}_D.$$

**Theorem (Auvray '17)**

*If $X \setminus D$ have a cscK metric of Poincaré type, then $D$ admits a cscK metric in $[\theta_X|_D]$.*

**Definition**

$$\omega_0 := \theta_X - \frac{2}{\underline{S}_D - \underline{S}} \sqrt{-1}\partial\overline{\partial} \log t$$

is a Kähler metric of Poincaré type, where $\theta_D = \theta_X|_D$ is a cscK metric.

# Asymptotic behavior of a cscK metric of Poincaré type

Assume that

$$\omega_0^{cscK} := \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_{cscK}$$

is a cscK metric of Poincaré type (Background Poincaré metric).

**Theorem (Auvray '17)**

*There exists $\delta > 0$ such that*

$$\varphi_{cscK} = O(t^{-\delta}) = O((\log \|\sigma_D\|^{-2})^{-\delta})$$

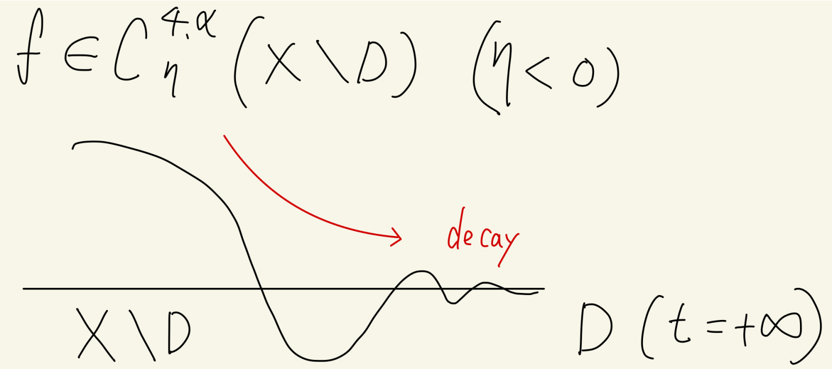
*as $t \to \infty$ at any differential order.*

# Function spaces

### Definition (Cheng-Yau, Kobayashi, Auvray)

We can define the Hölder space $C^{k,\alpha}(X \setminus D)$ (Cheng-Yau, Kobayashi) and the **weighted Hölder space** $C^{k,\alpha}_\eta(X \setminus D)$ (Auvray) for $\eta \in \mathbb{R}$ by

$$C^{k,\alpha}_\eta = C^{k,\alpha}_\eta(X \setminus D) := \{f \in C^{k,\alpha}(X \setminus D) \mid \|t^{-\eta} f\|_{C^{k,\alpha}(X \setminus D)} < \infty\}.$$

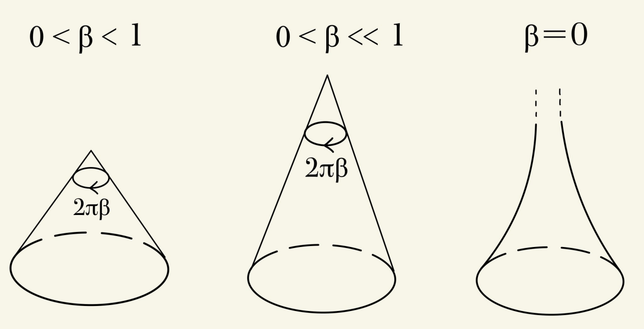


# Kähler cone metric

## Definition (Kähler metrics with cone singularities)

$\omega$ is a **Kähler cone metric of angle** $2\pi\beta$ iff it is quasi-isometric to the model cone metric near $D = \{z^1 = 0\}$:

$$\frac{\beta^2\sqrt{-1}dz^1 \wedge d\overline{z}^1}{|z^1|^{2(1-\beta)}} + \sum_j \sqrt{-1}dz^j \wedge d\overline{z}^j.$$



# cscK cone metric

- $c_1(X, D, \beta) := c_1(X) - (1-\beta)c_1(D)$
- $\theta \in c_1(X, D, \beta)$ : a smooth representation
- $f_0 \in C^\infty(X)$ s.t. $\mathrm{Ric}(\theta_X) = \theta + (1-\beta)\theta_X + \sqrt{-1}\partial\overline{\partial} f_0$
- $\omega_\theta$ : a solution of the following equation

$$\mathrm{Ric}(\omega_\theta) = \theta + 2\pi(1-\beta)[D] \iff \omega_\theta^n = e^{f_0}\|\sigma_D\|_{h_X}^{2\beta-2}\theta_X^n$$

## Definition (cscK cone metrics (Zheng))

$$\omega_{cscK}^n = e^F \omega_\theta^n, \quad \Delta_{\omega_{cscK}} F = \mathrm{tr}_{\omega_{cscK}}\theta - \underline{S}_\beta$$

Here,

$$\underline{S}_\beta := \frac{nc_1(X, D, \beta) \cup c_1(L_X)^{n-1}}{c_1(L_X)^n}.$$

## Lemma

*$\omega_\beta$ is a cscK cone metric of angle $2\pi\beta$ $\iff$ $S(\omega_\beta) = \underline{S}_\beta$ on $X \setminus D$.*

# Kähler potential of cone metric

## Definition (potential function of cone metric)

$$G_\beta(t) := \frac{2}{\underline{S}_{D,\beta} - \underline{S}_\beta} \int_2^t \frac{\beta}{e^{\beta y} - 1} dy.$$

Here,

$$\underline{S}_{D,\beta} := \frac{(n-1)c_1(X, D, \beta)|_D \cup c_1(L_X|_D)^{n-2}}{c_1(L_X|_D)^{n-1}}.$$

- $G_\beta(t) \to \dfrac{2}{\underline{S}_D - \underline{S}} \log t$ as $\beta \to 0$.
- $-\sqrt{-1}\partial\overline{\partial} G_\beta(t) \approx \dfrac{2}{\underline{S}_{D,\beta} - \underline{S}_\beta} \left( \dfrac{\beta}{1 - |z^1|^{2\beta}} \right)^2 e^{\beta a} \dfrac{\sqrt{-1} dz^1 \wedge d\overline{z}^1}{|z^1|^{2(1-\beta)}}.$

Here, we write $t = \log |z^1|^{-2} e^{-a}$ near $D = \{z^1 = 0\}$.

# Background Kähler cone metric

**Definition (Background Kähler cone metric)**

$$\omega_\beta := \theta_X - \sqrt{-1}\partial\overline{\partial}G_\beta(t) + \sqrt{-1}\partial\overline{\partial}\varphi_{cscK}$$

Since $\varphi_{cscK}$ decays near $D$,

$$\omega_\beta \approx \theta_X + \frac{2}{\underline{S}_{D,\beta} - \underline{S}_\beta}\left(\frac{\beta}{1-|z^1|^{2\beta}}\right)^2 e^{\beta a}\frac{\beta^2\sqrt{-1}dz^1\wedge d\overline{z}^1}{|z^1|^{2(1-\beta)}}.$$

**Remark**

Note that $\omega_\beta \to \omega_0^{cscK}$ as $\beta \to 0$. In general, $\omega_\beta$ is not a cscK cone metric.

# Fixed point formula and the Lichnerowicz operator

Consider the expansion : $S(\omega_\beta + \sqrt{-1}\partial\overline{\partial}\phi) = S(\omega_\beta) + L_{\omega_\beta}(\phi) + Q_{\omega_\beta}(\phi)$.
Here, $L_{\omega_\beta} : C^{4,\alpha}_\eta \to C^{0,\alpha}_\eta$ is the linearization of the scalar curvature operator. Then, we can write as

$$
\begin{aligned}
& S(\omega_\beta + \sqrt{-1}\partial\overline{\partial}\phi) = \underline{S}_\beta \\
& \qquad \Longleftrightarrow \quad \phi = -L^{-1}_{\omega_\beta}\left(S(\omega_\beta) - \underline{S}_\beta + Q_{\omega_\beta}(\phi)\right), \quad \phi \in C^{4,\alpha}_\eta(X \setminus D).
\end{aligned}
$$

**Problem**

*The map* $\phi \mapsto -L^{-1}_{\omega_\beta}\left(S(\omega_\beta) - \underline{S}_\beta + Q_{\omega_\beta}(\phi)\right)$ *have a fixed point in* $C^{4,\alpha}_\eta$*?*

$$
L_{\omega_\beta} = -\mathcal{D}^*_{\omega_\beta}\mathcal{D}_{\omega_\beta} + < \nabla^{1,0}S(\omega_\beta), \nabla^{0,1} * > .
$$

The first term $\mathcal{D}^*_{\omega_\beta}\mathcal{D}_{\omega_\beta}$ is called the **Lichnerowicz operator**.
$\mathcal{D}_{\omega_\beta} = \overline{\partial} \circ \nabla^{1,0}$, so $\mathrm{Ker}(\mathcal{D}^*_{\omega_\beta}\mathcal{D}_{\omega_\beta}) \simeq \{\text{holomorphic vector field}\}$.

# Sektnan's work

## Proposition (Sektnan '18)

*Assume that* $H^0(D, TD) = 0$ *and* $\mathrm{Aut}_0((X, L_X); D)$ *is trivial. There exists* $\kappa < 0$ *such that the Lichnerwicz operator*

$$\mathcal{D}^*_{\omega_0^{cscK}} \mathcal{D}_{\omega_0^{cscK}} : C^{4,\alpha}_\eta(X \setminus D) \to C^{0,\alpha}_\eta(X \setminus D)$$

*is isomorphic for any* $\eta \in (\kappa, 0)$*.*

## Remark

Sektnan showed more general result in the study of extremal Kähler metrics of Poincaré type. (He doesn't assume that $H^0(D, TD) = 0$ and $\mathrm{Aut}_0((X, L_X); D)$ is trivial.)

# Outline of the proof ($\exists$ cscK cone metric)

**Lemma**

$\exists \epsilon > 0$ *s.t.* $\|S(\omega_\beta) - \underline{S}_\beta\|_{C^{0,\alpha}_\eta} = O\left((-\log \beta)^{-\epsilon}\right).$

**Lemma**

*There exists* $K > 0$ *such that*

$$\|L_{\omega_\beta}\phi\|_{C^{0,\alpha}_\eta} \geq K\|\phi\|_{C^{4,\alpha}_\eta}, \quad 0 < \forall\beta \ll 1, \ \forall\phi \in C^{4,\alpha}_\eta.$$

**Proof.**

For small $\beta > 0$, the map

$$\phi_\beta \mapsto -L^{-1}_{\omega_\beta}\left(S(\omega_\beta) - \underline{S}_\beta + Q_{\omega_\beta}(\phi_\beta)\right)$$

is a contraction on a small ball of radius $r_\beta = O\left((-\log \beta)^{-\epsilon}\right)$ centered at $0 \in C^{4,\alpha}_\eta(X \setminus D)$. Thus, $\omega_\beta + \sqrt{-1}\partial\overline{\partial}\phi_\beta$ is a cscK cone metric and converges to $\omega_0^{cscK}$ as $\phi_\beta \to 0$ $(\beta \to 0)$. □

# 3. log K-stability (Algebraic Geometry)

# log test configuration

## Definition

A **log test configuration** $(\mathcal{X}, \mathcal{L}_{\mathcal{X}}, \mathcal{D})$ for $((X, L_X); D)$ is

1. $\mathcal{X}$: normal variety, $\pi : (\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \to \mathbb{C}$ : flat projective family with an equivariant $\mathbb{C}^*$-action,
2. $\pi^{-1}(1) \simeq (X, L_X)$,
3. $\mathcal{D}$ is the closure of $\mathbb{C}^*$-orbit of $D \in \pi^{-1}(1)$.

$\mathcal{X}_0 := \pi^{-1}(0)$ : central fiber of $\mathcal{X}$, $\quad \mathcal{D}_0$ : central fiber of $\mathcal{D}$

- $(\mathcal{X}, \mathcal{L}_{\mathcal{X}}, \mathcal{D})$ is product if $\mathcal{X} \simeq X \times \mathbb{C}, \mathcal{D} \simeq D \times \mathbb{C}$.
- $(\mathcal{X}, \mathcal{L}_{\mathcal{X}}, \mathcal{D})$ is trivial if it is product and $\mathbb{C}^*$-action is trivial.

# log Donaldson-Futaki invariant

- $d_k := \dim H^0(\mathcal{X}_0, \mathcal{L}^k|_{\mathcal{X}_0})$
- $\tilde{d}_k := \dim H^0(\mathcal{D}_0, \mathcal{L}^k|_{\mathcal{D}_0})$
- $w_k :=$ the total weight of the $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}^k|_{\mathcal{X}_0})$
- $\tilde{w}_k :=$ the total weight of the $\mathbb{C}^*$-action on $H^0(\mathcal{D}_0, \mathcal{L}^k|_{\mathcal{D}_0})$

We have the following formulae for sufficiently large $k$:

$$d_k = a_0 k^n + a_1 k^{n-1} + \ldots, \quad w_k = b_0 k^{n+1} + b_1 k^n + \ldots$$

$$\tilde{d}_k = \tilde{a}_0 k^{n-1} + \tilde{a}_1 k^{n-2} + \ldots, \quad \tilde{w}_k = \tilde{b}_0 k^n + \tilde{b}_1 k^{n-1} + \ldots$$

**Definition (log Donaldson-Futaki invariant)**

$$DF(\mathcal{X}, \mathcal{L}_{\mathcal{X}}, \mathcal{D}, \beta) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0} + (1-\beta)\frac{a_0 \tilde{b}_0 - \tilde{a}_0 b_0}{a_0}$$

# log K-stability

## Definition (log K-(semi)stability)

- $((X, L_X); D)$ is **log K-semistable with angle** $2\pi\beta$, if $DF(\mathcal{X}, \mathcal{L}_{\mathcal{X}}, \mathcal{D}, \beta) \geq 0$ for any log test configuration $((\mathcal{X}, \mathcal{L}_{\mathcal{X}}); \mathcal{D})$.
- $((X, L_X); D)$ is **log K-stable with angle** $2\pi\beta$ if it is log K-semistable and $DF(\mathcal{X}, \mathcal{L}_{\mathcal{X}}, \mathcal{D}, \beta) = 0$ iff $((\mathcal{X}, \mathcal{L}_{\mathcal{X}}); \mathcal{D})$ is trivial.

## Definition (uniform log K-stability)

- $((X, L_X); D)$ is **uniformly log K-stable with angle** $2\pi\beta$, if there is $\epsilon > 0$ s.t.

$$DF(\mathcal{X}, \mathcal{L}_{\mathcal{X}}, \mathcal{D}, \beta) \geq \epsilon \|(\mathcal{X}, \mathcal{L}_{\mathcal{X}})\|_m$$

for any log test configuration $((\mathcal{X}, \mathcal{L}_{\mathcal{X}}); \mathcal{D})$.

$\|(\mathcal{X}, \mathcal{L}_{\mathcal{X}})\|_m$ : Dervan's minimum norm of $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$

# log YTD conjecture

We assume that $\mathrm{Aut}_0((X, L_X); D)$ is trivial.

**Conjecture (log Yau-Tian-Donaldson conjecture)**

The existence of cscK cone metric of angle $2\pi\beta$ is equivalent to (uniform) log K-stability with angle $2\pi\beta$.

**Theorem (A-Hashimoto-Zheng '21)**

*If $((X, L_X); D)$ admits a cscK cone metric for angle $2\pi\beta$, then it is uniformly log K-stable with angle $2\pi\beta$.*

**Corollary (A. '22)**

*Assume that $H^0(TD) = 0$ and $\mathrm{Aut}_0((X, L_X); D)$ is trivial. If $X \setminus D$ has a cscK metric of Poincaré type, then $((X, L_X); D)$ is uniformly log K-stable with sufficiently small angle $2\pi\beta$.*

# log K-semistability with angle 0 ($D$ : cscK)

**Conjecture (J.Sun-S.Sun '16)**

Assume that $\underline{S}_D \leq 0$. If $(D, L_X|_D)$ has a cscK metric, then the pair $((X, L_X); D)$ is log K-semistable with cone angle 0.

**Theorem (S. Sun '13)**

*Assume that $\underline{S}_D = 0$. If $(D, L_X|_D)$ has a cscK metric, the pair $((X, L_X); D)$ is* **strictly** *log K-semistable with cone angle 0. (Namely, it is log K-semistable and there exists a nontrivial log test configuration with vanishing log Donaldson-Futaki invariant.)*

# log K-semistability with angle 0 ($X \setminus D$ : cscK)

**Conjecture (Székelyhidi '06)**

$X \setminus D$ admits a cscK metric of Poincaré type iff $((X, L_X); D)$ is log K-stable with angle 0 and $\underline{S} < \underline{S}_D$.

**Corollary (A. '22)**

*Assume that* $H^0(TD) = 0$ *and* $\mathrm{Aut}_0((X, L_X); D)$ *is trivial. If* $X \setminus D$ *has a cscK metric of Poincaré type, then* $((X, L_X); D)$ *is log K-semistable with angle* $0$.

$X \setminus D$ : Poincaré type cscK $\xrightarrow{\text{Auvray}}$ $D$ : cscK s.t. $\underline{S}_D < 0$

$X \setminus D$ : Poincaré type cscK $\Longleftrightarrow^{\mathrm{Conj(Sz)}}$ $((X, L_X); D, 0)$ : log K-stable

$X \setminus D$ : Poincaré type cscK $\xRightarrow{\text{A.}}$ $((X, L_X); D, 0)$ : log K-semistable

$D$ : cscK s.t. $\underline{S}_D < 0$ $\Longrightarrow^{\mathrm{Conj(SS)}}$ $((X, L_X); D, 0)$ : log K-semistable

$((X, L_X); D, 0)$ : log K-stable $\Longrightarrow$ $((X, L_X); D, 0)$ : log K-semistable

Thank you for your attention !