Quaternionic k-vector fields on Quaternionic Kähler mfds

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§Intro

Twistor theory

$\dim_{\mathbb{R}}=4$ (Penrose, Atiyah-Hitchin-Singer)

(M^4, g) : self-dual $\Longleftrightarrow$ (Z^3, J) : complex mfd

$$\begin{array}{c} Z \\ \downarrow \mathbb{CP}^1 \\ M \end{array}$$

twistor space

g : Einstein $\Longleftrightarrow$ η : holo. $(1,0)$-form (valued with $K^{\frac{1}{2}}_Z$)

$\dim_{\mathbb{R}} \geq 8$ (Salamon)

(M^{4n}, g) : Quaternionic Kähler ($\Rightarrow$ Einstein) $s>0, s=0, s<0$

$\Longleftrightarrow$ (Z^{2n+1}, J) : cpx mfd

twistor sp.

s : Scalar curv. of g.

$s=0 \Rightarrow Z \underset{\text{diff.}}{\cong} M \times \mathbb{CP}^1$, $Z \underset{\text{holo.}}{\ncong} M \times \mathbb{CP}^1$

$s \neq 0 \Rightarrow Z$: cpx contact mfd

$s>0 \Rightarrow Z$: positive Kähler-Einstein

Geometric data in Riem. mfd M

$\longleftrightarrow$ holomorphic data in cpx mfd Z.

Q. What is a k-vector field on M corresponding to a holomorphic $(k,0)$-vector field on Z?

On cpx mfd,

holo. vector field (automorphism)
holo. Poisson str
⋮

$\Bigg\}$ holo. $(k,0)$-vector field

Thm X : quaternionic k-vector field on M

$\underset{1:1}{\longleftrightarrow}$ X' : holo. $(k,0)$-vector field on Z

§ Quaternionic Kähler mfds.

(M^{4n}, g) : Riem. mfd of $\dim_{\mathbb{R}} 4n$

Def (M^{4n}, g) : quaternionic Kähler mfd $(n>1)$

$\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Hol}_g \subset Sp(n)\cdot Sp(1) \left(:= Sp(n)\times Sp(1)/\mathbb{Z}_2\right)$

(M^4, g) : quaternionic Kähler $(n=1)$

$\overset{\text{def}}{\Longleftrightarrow}$ self-dual and Einstein

(M, g) : q.K. $\Rightarrow$ Einstein (s : scalar curv.)

$s>0$: Wolf space (cpt symmetric sp.) + $\underset{\sim}{?}$

S^4, $\overline{\mathbb{C}P^2}$

$s<0$: Alekseevsky sp. (non-cpt symm. sp. $\xrightarrow{/\Gamma}$ cpt. , homog. (non symm.)) + $\underset{\sim}{?}$

$s=0$: (locally) hyper Kähler

§ Twistor sp.

(M^{4n}, g) : q.K. $\Rightarrow$ ${}^{\exists}F \subset P(TM)$: $Sp(n)\cdot Sp(1)$-bdle.

$$\begin{array}{ccccc} Sp(n)\cdot Sp(1) & \curvearrowright & \mathbb{H}^n & \longrightarrow & \mathbb{H}^n \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [A, g] & & h & \longmapsto & A h g^{-1} \end{array}$$

$\Rightarrow$ $TM = F \times_{Sp(n)\cdot Sp(1)} \mathbb{H}^n$

$Q := F \times_{Sp(n)\cdot Sp(1)} \langle i, j, k \rangle$ $\quad \left(\mathbb{H}^n \curvearrowleft \underset{\langle 1, i, j, k \rangle}{\overset{\shortparallel}{\mathbb{H}}} \right)$

(: quaternionic str)

$\subset End(TM)$

$Z = \{ A \in Q \mid A^2 = -id \}$: Twistor sp.

$$\begin{array}{l} \downarrow S^2 \\ M \end{array} \quad \overset{\shortparallel}{\{} aI_- + bJ_- + cK_- \mid a^2 + b^2 + c^2 = 1 \}$$

$$\left(\begin{array}{lll} \mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k & \mathbb{R}^4 & \mathbb{C} + j\mathbb{C} \simeq \mathbb{C}^2 \\ \curvearrowleft i, j, k & I_-, J_-, K_- & \curvearrowleft \mathbb{C},\ j_0 : \mathbb{C}^2 \to \mathbb{C}^2 : \text{anti-}\mathbb{C}\text{-linear} \\ ji = -ij \quad \overset{\shortparallel}{} ij & & j_0^2 = -id. \end{array} \right)$$

E-H formalizm

$E := F \times_{\rho_+} \mathbb{H}^n$, $\rho_+([A, g])h = Ah$ $\quad$ $Sp(n) \cdot Sp(1)$

$H := F \times_{\rho_-} \mathbb{H}$, $\rho_-([A, g])h = hg^{-1}$

(is not globally defined) $\quad \overset{n}{\otimes} E \otimes \overset{m}{\otimes} H$: global) if $n+m =$ even

$\Rightarrow TM = E \otimes_{\mathbb{H}} H \qquad (\mathbb{H}^n = \mathbb{H}^n \otimes_{\mathbb{H}} \mathbb{H})$

$P(H^*)$: frame b'dle of H^* (as right $\mathbb{H}$-module b'dle)

$\Rightarrow Z = P(H^*) / GL(1, \mathbb{C})$

$$\begin{tikzcd} P(H^*) \arrow[d, "GL(1,\mathbb{H})"'] \arrow[dr, "GL(1,\mathbb{C})"] & \\ M & Z \arrow[l, "\mathbb{C}P^1"] \end{tikzcd}$$

$\mathbb{H}^n \simeq \mathbb{C}^{2n}$

E, H : $\mathbb{C}$-vector b'dle. with J_E, J_H

$\Rightarrow TM \otimes \mathbb{C} = E \otimes_{\mathbb{C}} H \qquad (u^* \longleftrightarrow u \longleftrightarrow \{u, ju\})$

$u^* \in P(H^*) \rightsquigarrow$ almost cpx str on M s.t.

$cu^* \rightsquigarrow$ $\underline{T_x^{1,0} M = E \otimes_{\mathbb{C}} \langle u \rangle}$, $\underline{T_x^{0,1} M = E \otimes_{\mathbb{C}} \langle ju \rangle}$

$Z = P(H^*) / \mathbb{C}^*$

$T_u P(H^*) = \mathcal{H} \oplus \mathcal{V} \leftarrow \mathbb{H} = \mathbb{C}^2$

(integrable) Cpx str. on $P(H^*) \rightsquigarrow$ cpx str. on Z

§ quaternionic k-vector fields

cpx symp str

E, H : $\mathbb{C}$-vector b'dle. with $J_E, J_H, \omega_E, \omega_H$

∇ : Levi-Civita conn. of S $\rightsquigarrow$ conn. of E, H.

$$\nabla : \Lambda^k E\, S^m H \longrightarrow \Lambda^k E\cdot E^*\ \underbrace{S^m H\cdot H^*}_{= S^{m+1}H \oplus S^{m-1}H} \qquad H^* \xrightarrow{\omega_H} H$$

$\mathscr{D}_{\Lambda^k E}$: twistor op. (Penrose op.) $\quad \Lambda^k E\, S^m H \to S^{m+1}H$

$D_{\Lambda^k E}$: Dirac op (Dirac-Fueter op) $\quad \Lambda^k E\, S^m H \to S^{m-1}H$

※ $k=0, m=1$

$$D : H \longrightarrow E^* \simeq E$$

$$\mathscr{D} : H \longrightarrow E^* S^2 H \simeq E S^2 H$$

$m=k$

$$\Lambda^k E \otimes S^k H \subset \Lambda^k TM \otimes \mathbb{C} \quad : k\text{-vector field}$$

$$TM \otimes \mathbb{C} = EH$$

$$\Lambda^2 TM \otimes \mathbb{C} = \Lambda^2 E S^2 H \oplus S^2 E \Lambda^2 H$$

$n=1$ $\qquad$ self-dual $\qquad$ anti-

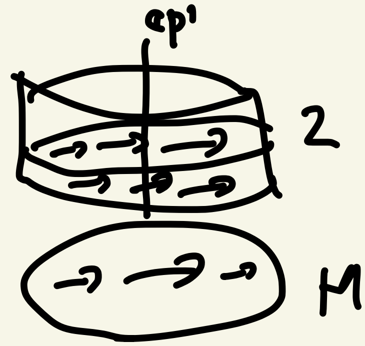
$$\Lambda^2 T^*M = \Lambda^2 E^* S^2 H^* \oplus S^2 E^* \Lambda^2 H^* \simeq S^2 H^* \oplus S^2 E^*$$

$$\mathcal{D}_{\Lambda^k E} : \Lambda^k E S^k H \longrightarrow \underbrace{\Lambda^k E \cdot E^*}_{=\ \underset{\text{trace less}}{(\Lambda^k E \cdot E^*)_0} \oplus \Lambda^{k-1} E} S^{k+1} H$$

<u>Fact</u> $X \in \ker \mathcal{D}_{\Lambda^k E}$

$\Longleftrightarrow$ $\hat{X}_h^{k,0}$: holo. $(k,0)$-v.f. on Z

($\hat{X}_h$: horizontal lift to Z)



$$\mathcal{D}^0_{\Lambda^k E} : \Lambda^k E S^k H \longrightarrow (\Lambda^k E \cdot E^*)_0 S^{k+1} H$$

<u>Def</u> $X \in A^0(\Lambda^k E S^k H)$: quaternionic k-vector field

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \mathcal{D}^0_{\Lambda^k E}(X) = 0 & (1 \leq k \leq 2n-1) \\ \mathcal{D}_{\Lambda^{k-1} E} \circ \mathrm{tr} \circ \mathcal{D}_{\Lambda^k E}(X) = 0 & (k = 2n) \end{cases}$$

※ $k = 2n \Rightarrow (\Lambda^{2n} E \cdot E^*)_0 = \{0\} \Rightarrow \mathcal{D}^0_{\Lambda^{2n} E} = 0$

$$\begin{tikzcd}
\Lambda^k E S^k H \arrow[r, "\mathcal{D}_{\Lambda^k E}"] & \Lambda^k E E^* S^{k+1} H \arrow[d, "\mathrm{tr}"] & \\
& \Lambda^{k-1} E S^{k+1} H \arrow[r, "\mathcal{D}_{\Lambda^{k-1} E}"] & \Lambda^{k-1} E \cdot E^* S^{k+2} H
\end{tikzcd}$$

example $X \in A^0(EH)$: quaternionic 1-v. f.

$\Leftrightarrow$ X : v. f preserving Q (i.e. $L_X Q \subset Q$)

($\rightsquigarrow$ diffeomorphism preserving Q)

§ Main theorem

$$Q(\overset{k}{\Lambda} E S^k H) := \{ X \in A^0(\overset{k}{\Lambda} E S^k H) : \text{quaternionic} \}$$

Thm 1 $Q(\overset{k}{\Lambda} E S^k H) \simeq \mathcal{O}(\overset{k}{\Lambda} T^{1,0} Z)$

$$\begin{array}{ccc} \cup & & \Psi \\ X & \mapsto & \hat{X}_h^{k,0} + {}^{\exists} Y \wedge \nu \end{array}$$

$$\left(\begin{array}{l} \nu : l^{-2}\text{-valued vertical v. f} \\ Y : l^{2}\text{-valued horizontal } (k-1) \text{ v.f.} \\ l \to Z : \text{line bdle} \quad \text{s.t. } l|_{\mathbb{CP}^1} = \mathcal{O}(1) \end{array} \right)$$

§ Proof of Main thm

lift

$u \in P(H^*) \rightsquigarrow \{u^m, u^{m-1}(ju), \dots, (ju)^m\}$: basis of $S^m H$

symmetrization of $u \otimes \cdots \otimes u \otimes ju$

$\longleftrightarrow \{1^m, 1^{m-1}j, \dots, j^m\}$: basis of $S^m \mathbb{H}$

<u>M</u> $\qquad$ <u>$P(H^*)$</u> $\qquad$ <u>Z</u>

$\tilde{A}^q(\Lambda^k E S^m \mathbb{H})$ ← pull-back by $P(H^*) \to M$.

$$\tilde{\xi} = \tilde{\xi}_0 1^m + \tilde{\xi}_1 1^{m-1} j + \dots + \tilde{\xi}_m j^m$$

$$\left(R_a^* \tilde{\xi} = \rho_{S^m \mathbb{H}}(a) \tilde{\xi}, \quad a \in GL(1, \mathbb{H}) \right)$$

$A^q(\Lambda^k E S^m H) \ni$

$$\xi = \hat{\xi}_0 u^m + \hat{\xi}_1 u^{m-1}(ju) + \dots + \hat{\xi}_m (ju)^m \longleftrightarrow \tilde{\xi}$$

$$\tilde{\xi} \mapsto \tilde{\xi}_0 \in \tilde{A}^q_{(m,0)}(\Lambda^k E)$$

$\mathbb{C}^*$-order $(m-i, i)$

$\Longleftrightarrow R_c^* \alpha = c^{m-i} \bar{c}^i \alpha$

$c \in \mathbb{C}^*$

$$\xi \xrightarrow{\text{lift}} \hat{\xi}_0 \in \hat{A}^q(\Lambda^k E \otimes l^m) \longleftrightarrow \tilde{\xi}_0$$

pull-back by $Z \to M$.

$$\theta := \mathrm{id}_{TM} \in A^1(E_{\overset{\shortparallel}{TM}}H) \rightsquigarrow \hat{\theta} = \underbrace{\tilde{\theta}_0}\, 1 + \underbrace{\tilde{\theta}_1}\, j \in \hat{A}^1(EH)$$

$$\rightsquigarrow \hat{\theta}_0 \in \hat{A}^1(E\otimes l) \qquad \hat{\theta}_1 \in \hat{A}^1(E\otimes \bar{l})$$

$\Rightarrow \hat{\theta}_0$: (1,0)-form, $\hat{\theta}_1$: (0,1)-form.

$\overset{\shortparallel}{\ } \sum e_i \otimes \alpha^i \rightsquigarrow \langle \alpha^1, \cdots, \alpha^{2n} \rangle \simeq T^{1,0}M$

$$\hat{\theta}_0^k := \hat{\theta}_0 \wedge \hat{\theta}_0 \wedge \cdots \wedge \hat{\theta}_0 \in \hat{A}^k(\overset{k}{\wedge} E \otimes l^k)$$

$$= \sum e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \rightsquigarrow \langle \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \rangle \simeq \overset{k}{\wedge} T^{1,0}M.$$

$$X \in A^0(\overset{k}{\wedge} E\, S^k H) \rightsquigarrow \tilde{X} \in \hat{A}^0(\overset{k}{\wedge} E\, S^k H)$$

$$\overset{\shortparallel}{\ } \underbrace{\tilde{x}_0 u^k}_{(k,0)} + \underbrace{\tilde{x}_1 u^{k}\cdot ju}_{(k-1,1)} + \cdots + \tilde{x}_k (ju)^k \qquad \overset{\shortparallel}{\ } \tilde{X}_0 1^k + \tilde{X}_1 1^{k-1} j + \cdots + \tilde{X}_k j^k$$

$$\downarrow$$

$$\hat{X}_0 \in \hat{A}^0(\overset{k}{\wedge} E \otimes l^k)$$

($\because$ $T^{1,0}M = E\otimes u$, $T^{0,1}M = E \otimes ju$)

Prop 1 $X \in A^0(\overset{k}{\wedge} E\, S^k H)$, $Y \in A^0(\overset{k-1}{\wedge} E\, S^{k+1} H)$ $\quad \hat{Y}_0 \in \hat{A}^0(\overset{k-1}{\wedge} E \otimes l^{k+1})$

X : q-v.f. and $Y = \mathrm{tr} \circ \mathcal{D}_{\overset{k}{\wedge} E}(X)$

$$\Leftrightarrow \begin{cases} \bar{\partial} \hat{X}_0 - \hat{Y}_0 \wedge_E \hat{\theta}_1 = 0 & (k \neq 2n) \\ \qquad \text{〃} \qquad , \ \bar{\partial} \hat{Y}_0 = 0 & (k = 2n) \end{cases}$$

($\bar{\partial} := (\nabla_{E \otimes l})^{0,1}$)

Holomorphic $(k,0)$-vector fields on Z

A : conn. form of $P(\mathbb{H}^*)$

$\|$ ($sp(1)$-valued) $\subset \mathbb{H} = \mathbb{C} + j\mathbb{C}$

$\eta_0 + j\eta_1$

$$\Rightarrow \quad TP(\mathbb{H}^*) = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{V}}, \quad \tilde{\mathcal{H}} := \ker A$$

$$\langle \eta_0, \eta_1 \rangle = (\tilde{\mathcal{V}}^*)^{1,0}$$

ν_1 : $(1,0)$-part of fundamental v.f. $\hat{j}$

$(\eta_1(\nu_1) = 1)$

$$\eta \in A_Z^{1,0}(\ell^2), \quad \nu \in A^0(\ell^{-2} \otimes T^{1,0}Z)$$

$(\eta(\nu) = 1)$

$$\Rightarrow \quad TZ = \hat{\mathcal{H}} \oplus \hat{\mathcal{V}}, \quad \hat{\mathcal{H}}^{1,0} := \ker \eta$$

$$\bigwedge^k T^{1,0}Z = \bigwedge^k \hat{\mathcal{H}}^{1,0} \oplus \bigwedge^{k-1} \hat{\mathcal{H}}^{1,0} \wedge \hat{\mathcal{V}}^{1,0}$$

$\left(\begin{array}{l} P(\mathbb{H}^*) = P(\ell^{-1}) : \text{frame bdle of } \ell^{-1} \\ \quad \downarrow \\ \quad Z \qquad \eta_0 : \text{conn. form.} \end{array} \right)$

$X' \in A^0(\overset{k}{\wedge} T^{1,0}Z) \Rightarrow X' = X'_h + Y' \wedge \nu$

Prop 2 X' : holo. $(k,0)$-v.f on Z $\Leftrightarrow \nabla^{0,1}X' = 0 \Leftrightarrow \begin{cases} \hat\theta_0^k(\nabla^{0,1}X') = 0 \\ \hat\theta_0^{k-1} \wedge \eta(\nabla^{0,1}X') = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} \bar\partial(\hat\theta_0^k(X'_h)) - \hat\theta_0^{k-1}(Y') \wedge_E \hat\theta_1 = 0 & (k \neq 2n) \\ \qquad \text{"} \qquad , \quad \bar\partial(\hat\theta_0^{k-1}(Y')) = 0 & (k = 2n) \end{cases}$$

lift $\hat X_0 \longleftrightarrow$ horizontal lift $\hat X_h$ $\quad (\hat\theta_0^k \in \hat A^k(\overset{k}{\wedge} E \otimes \ell^k))$

$$\begin{array}{ccc} X \in A^0(\overset{k}{\wedge} E S^k H) & \longleftrightarrow & \hat X_0 \in \hat A^0(\overset{k}{\wedge} E \otimes \ell^k) \\ & \searrow\!\!\nwarrow & \updownarrow \hat\theta_0^k \\ & \hat X_h^{k,0} \in A^0(\overset{k}{\wedge} \hat H^{1,0}) & \end{array}$$

$$\begin{array}{ccc} Y \in A^0(\overset{k-1}{\wedge} E S^{k+1} H) & \longleftrightarrow & \hat Y_0 \in \hat A^0(\overset{k-1}{\wedge} E \otimes \ell^{k+1}) \\ & & \updownarrow \hat\theta_0^{k-1} \\ & & \hat Y_h \in A^0(\overset{k-1}{\wedge} \hat H^{1,0} \otimes \ell^2) \end{array}$$

$\Rightarrow$ X : quaternionic, $Y = \mathrm{tr} \circ D_{\wedge E}(X)$ $\overset{\text{Prop 1}}{\Longleftrightarrow}$ $\bar\partial \hat X_0 - \hat Y_0 \wedge_E \hat\theta_1 = 0$

$\Leftrightarrow$ $\hat X_h^{k,0} + \hat Y_h \wedge \nu$: holo. $\overset{\text{prop 2}}{\Longleftrightarrow}$ $\hat\theta_0^k(\hat X_h^{k,0})$ $\hat\theta_0^{k-1}(\hat Y_h)$ $\square$

§ quaternionic real k-vector field

$\tau := \overset{k}{\otimes} J_E \otimes \overset{m}{\otimes} J_H \in \mathrm{End}(\overset{k}{\Lambda} E\, S^m H)$

$k+m$: even $\Rightarrow$ τ is real str.

τ preserves ∇, ω_H, tr $\rightsquigarrow$ $\mathcal{D}_{\overset{k}{\Lambda}E}$, $\mathcal{D}^{\circ}_{\overset{k}{\Lambda}E}$

$\rightsquigarrow$ τ : real str on $Q(\overset{k}{\Lambda} E S^k H)$

$Q(\overset{k}{\Lambda} E S^k H)^{\tau} \subset A^0(\overset{k}{\Lambda} TM)$: real v. f.

($X \in \overset{k}{\Lambda} E S^k H \Rightarrow \tau(X) = \overline{X}$: cpx conjugate)

$\hat{\tau} : \mathcal{O}(\overset{k}{\Lambda} T^{1,0} Z) \to \mathcal{O}(\overset{k}{\Lambda} T^{1,0} Z)$ $\quad R_j \curvearrowright P(H^*)$

$X' \mapsto \hat{\tau}(X') := \overline{(R_{[j]})_* X'}$ $\quad R_{[j]} \curvearrowright Z$

Thm 2 $Q(\overset{k}{\Lambda} E S^k H)^{\tau} \simeq \mathcal{O}(\overset{k}{\Lambda} T^{1,0} Z)^{\hat{\tau}}$

<u>example</u> $M = \mathbb{H}P^1 (= S^4)$, $Z = \mathbb{C}P^3$

$$Q(E H) \simeq \mathfrak{gl}(4, \mathbb{C}) / \langle id \rangle$$

$$Q(E, H)^\tau \simeq \mathfrak{gl}(2, \mathbb{H}) / \langle id \rangle$$

$$Q(\Lambda^2 E S^2 H) \simeq S^2 \otimes \Lambda^2 (\otimes^2 \mathfrak{gl}(4, \mathbb{C})) / S^2 \otimes \Lambda^2 (\mathfrak{gl}(4, \mathbb{C}) \otimes id)$$

$$Q(\Lambda^2 E S^2 H)^\tau \simeq S^2 \otimes \Lambda^2 (\otimes^2 \mathfrak{gl}(2, \mathbb{H})) / S^2 \otimes \Lambda^2 (\mathfrak{gl}(2, \mathbb{H}) \otimes id)$$

$$\left(S^2 \otimes \Lambda^2 : \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \otimes^2 \mathfrak{gl}(4, \mathbb{C})}}{\otimes^2 \mathbb{C}^4 \otimes \otimes^2 (\mathbb{C}^4)^*} \longrightarrow S^2 \mathbb{C}^4 \otimes \Lambda^2 (\mathbb{C}^4)^* \right)$$

$$\Rightarrow \quad \dim_{\mathbb{C}} Q(E H) = 15$$

$$\dim_{\mathbb{C}} Q(\Lambda^2 E S^2 H) = 45$$

§ graded algebra $\bigoplus \mathcal{Q}(\overset{*}{\Lambda} E S^* H)$

Schouten-Nijenhuis bracket

$$[\ ,\ ] : A^0(\overset{k}{\Lambda} TM) \times A^0(\overset{k'}{\Lambda} TM) \longrightarrow A^0(\overset{k+k'-1}{\Lambda} TM) \ : \text{bilinear}$$

s.t. • $[X, X'] = (-1)^{kk'}[X', X]$

• $(-1)^{k(k''-1)}[[X, X'], X''] + (-1)^{k'(k-1)}[X', [X'', X]]$

$+ (-1)^{k''(k'-1)}[X'', [X, X']] = 0$

$$\left(\begin{array}{l} \text{On } Z \\ [\ ,\ ] : \mathcal{O}(\overset{k}{\Lambda} T^{1,0} Z) \times \mathcal{O}(\overset{k'}{\Lambda} T^{1,0} Z) \to \mathcal{O}(\overset{k+k'-1}{\Lambda} T^{1,0} Z) \end{array} \right)$$

$$X, X' \in A^0(EH) \Rightarrow [X, X'] \in A^0(EH)$$

In general,

$$X \in A^0(\overset{k}{\Lambda} E S^k H),\ X' \in A^0(\overset{k'}{\Lambda} E S^{k'} H)$$

$$\Rightarrow [X, X'] \in A^0(\overset{k''}{\Lambda} E S^{k''} H)$$

$$(k'' := k + k' - 1)$$

Def For $X \in A^0(\overset{k}{\Lambda} E S^k H)$, $X' \in A^0(\overset{k'}{\Lambda} E S^{k'} H)$,

$$[X, X']_Q := \overset{k''}{\Lambda} E S^{k''} H\text{-part of } [X, X']$$

Prop3 $X \in Q(\overset{k}{\Lambda} E S^k H)$, $X' \in Q(\overset{k'}{\Lambda} E S^{k'} H)$

$$\Rightarrow [X, X']_Q \in Q(\overset{k''}{\Lambda} E S^{k''} H)$$

$$\underbrace{[\hat{X}_h^{k,0} + Y \wedge v, \hat{X}'^{k',0}_h + Y' \wedge v]}_{\text{holo.}} = \underbrace{[\hat{X}_h^{k,0} + Y \wedge v, \hat{X}'^{k',0}_h + Y' \wedge v]_h}_{= \left(\underbrace{[X, X']_Q}_{\text{quaternionic}}\right)_h^{k.0}} + \bullet \wedge v$$

holo.

Thm3 $\left(\bigoplus_{k=1}^{2n} Q(\overset{k}{\Lambda} E S^k H), [\ ,\ ]_Q\right) \simeq \left(\bigoplus_{k=1}^{2n} \mathcal{O}(\overset{k}{\Lambda} T^{1,0} Z), [\ ,\ ]\right)$

$\underline{k=1}$

$(Q(EH), [\ ,\ ]) \simeq (\mathcal{O}(T^{1,0} Z), [\ ,\ ])$ as Lie algebra.