Perelman's entropy in Kähler geometry and its non-archimedean reflection

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Contents

This talk is based on my 4 articles:

- '19, Constant μ-scalar curvature Kähler metric ...
- '20, Equivariant calculus on μ-character and μK-stability ...
- '21a, Entropies in μ-framework of ..., I
- '21b, Entropies in μ-framework of ..., II (in preparation)

1. μ-cscK metrics and μK-stability

$$\mathrm{Ham}_T(M,\omega) \curvearrowright (\mathcal{J}_T(M,\omega), \Omega_\xi) \xrightarrow{\mathcal{S}^\lambda_\xi} \mathfrak{ham}_T^\vee$$

The story begins with "moment map picture"....

μ-cscK metric

(X, L): a compact Kähler manifold and an (integral) Kähler class L

For a Kähler metric ω, a real holomorphic vector field ξ is called *Hamiltonian* if $\exists \mu_\xi \in C^\infty(X)$ s.t. $d\mu_\xi = -i_\xi \omega$ ($\Leftrightarrow \nabla \mu_\xi = -J\xi \rightsquigarrow \mathcal{L}_\xi \omega = 0$).

Definition (μ-cscK metric, '19, cf. Y. Nakagawa, generalized KRs '11)

A Kähler metric $\omega \in L$ is called $\check{\mu}^\lambda_\xi$-cscK metric if ξ is Hamiltonian and

$$s^\lambda_{\mu_\xi}(\omega) := (s(\omega) + \bar{\square}\mu_\xi) + (\bar{\square}\mu_\xi + |\partial^\sharp \mu_\xi|^2) - \lambda\mu_\xi = \mathrm{const}.$$

- KRs $\mathrm{Ric}(\omega) - \mathcal{L}_{J\xi}\omega = \lambda\omega \Leftrightarrow \mu^\lambda_\xi = \check{\mu}^\lambda_{-2\xi}$-cscK metric in L: $\lambda L = 2\pi c_1(X)$.
- Extremal metric $\Leftrightarrow$ the limit of μ^λ-cscK metrics as $\lambda \to -\infty$.

(For each $\lambda \in \mathbb{R}$, there exists ξ_λ with $\mathrm{Fut}^\lambda_{\xi_\lambda} = 0$. As $\lambda \to -\infty$, $\lambda\xi_\lambda \to \xi_{\mathrm{ext}}$.)

On $(X, L) = (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, -K_X)$, there exists a μ^λ-cscK metric for every $\lambda \in \mathbb{R}$ which gives a smooth path from KRs at $\lambda = 2\pi$ to extremal metric at $\lambda = -\infty$.

$T = \overline{\exp \mathbb{R}\xi} \subset \mathrm{Isom}(X, \omega) \rightsquigarrow T$-equivariant formulation of μK-stability.

Test configuration

Let (X, L) be a T-equivariant polarized variety. A T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L) is a $\mathbb{C}^\times \times T$–equivariant flat family $\varpi : \mathcal{X} \to \mathbb{A}^1$ of schemes over $\mathbb{A}^1 = \mathbb{C}$ endowed with

- a relatively ample $\mathbb{C}^\times \times T$-equivariant $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ and
- a specific isomorphism $(X, L) \cong (\mathcal{X}_1, \mathcal{L}|_{\mathcal{X}_1})$ to the fibre over $1 \in \mathbb{A}^1$.

We have a natrual compactification $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ over $\mathbb{P}^1$.

We assign the following filtration $\mathcal{F}_{(\mathcal{X},\mathcal{L})}$ on $R = \bigoplus_m R_m = \bigoplus_m H^0(X, L^{\otimes m})$:

$$\mathcal{F}^\lambda_{(\mathcal{X},\mathcal{L})} R_m := \{s \in R_m \mid \varpi^{-\lceil \lambda \rceil} \bar{s} \text{ extends to a section of } \mathcal{L}^{\otimes m}\}.$$

For a *normal* test configuration $(\mathcal{X}, \mathcal{L})$ and an irreducible component $E \subset \mathcal{X}_0$, we assign the following variation v_E on $\mathbb{C}(X)$:

$$v_E(f) := \frac{\mathrm{ord}_E(f \circ p_X)}{\mathrm{ord}_E \mathcal{X}_0}$$

μK-stability

Let K_X denote the canonical sheaf on X, which is a T-equiv reflexive sheaf. For a T-equiv *normal* test configuration $(\mathcal{X}, \mathcal{L})$, we define $\check{\mathrm{Fut}}^\lambda_\xi(\mathcal{X}, \mathcal{L})$ by

$$\frac{(2\pi K^T_{\bar{\mathcal{X}}/\mathbb{P}^1} - \lambda \bar{\mathcal{L}}_T . e^{\bar{\mathcal{L}}_T}; \xi) \cdot (e^{L_T}; \xi) - (2\pi K^T_X - \lambda L_T . e^{L_T}; \xi) \cdot (e^{\bar{\mathcal{L}}_T}; \xi)}{(e^{L_T}; \xi)^2} - \lambda \frac{(e^{\bar{\mathcal{L}}_T}; \xi)}{(e^{L_T}; \xi)}.$$

Definition (μK-stability, '20)

A T-equivariant polarized normal variety (X, L) is called $\check{\mu}^\lambda_\xi$K-semistable if $\check{\mathrm{Fut}}^\lambda_\xi(\mathcal{X}, \mathcal{L}) \geq 0$ for every normal T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$.

Theorem (Lahdili '19, I. '20, cf. Apostolov–Lahdili–Jubert '21)

If a polarized manifold (X, L) admits a $\check{\mu}^\lambda_\xi$-cscK metric, it is $\check{\mu}^\lambda_\xi$K-semistable.

Boundedness + the slope of μ-Mabuchi functional along smooth subgeodesic ray subordinate to a smooth test configuration. It is equivariant localization.

You can safely forget the definition of μ-Futaki invariant! This is hard to compute. We have another criterion for μK-semistability as we will see.

Equivariant cohomology

Let X be a finite dimensional locally compact topological space with a continuous action of a torus $T \cong U(1)^{\times k}$ or $(\mathbb{C}^\times)^{\times k}$.

- Recall the equivariant cohomology of X is defined as

$$H_T^*(X, \mathbb{Z}) := H^*(X_T, \mathbb{Z}),$$

  using the Borel construction $X_T := X \times_T ET$ for the classifying space $ET \overset{\text{htpic}}{\approx} (\mathbb{C}^\infty \setminus \{0\})^{\times k}$.

- Equivariant locally finite (Borel–Moore) homology is defined as

$$H_p^{\mathrm{lf}, T}(X, \mathbb{Z}) := H^{\mathrm{lf}}_{\dim_{\mathbb{R}}(X_T^p/X)+p}(X_T^p, \mathbb{Z}),$$

  using $X_T^p := X \times_T E^p T$ for a finite dimensional approximation $E^p T := (\mathbb{C}^{l_p+1} \setminus \{0\})^{\times k}$ for large l_p: $2k(l_p + 1) > \dim_{\mathbb{R}} X - p + 1$.

Poincare duality: $H_T^p(X, \mathbb{Z}) \cong H^{\mathrm{lf}, T}_{\dim_{\mathbb{R}} X - p}(X, \mathbb{Z})$ for smooth oriented X.

Equivariant intersection 1

Let $X \circlearrowleft T$ be a compact normal complex space of $\dim_{\mathbb{C}} X = n$. Recall

$$H^{\mathrm{lf},T}_{2n-2}(X,\mathbb{Z}) \cong H^{\mathrm{lf},T}_{2n-2}(X^{\mathrm{reg}},\mathbb{Z}) \cong H^2_T(X^{\mathrm{reg}},\mathbb{Z}).$$

- For a T-equivariant complex line bundle L on X,

$$c_{1,T}(L) := c_1(L \times_T ET) \in H^2_T(X,\mathbb{Z}).$$

- For a T-equivariant reflexive sheaf D on X ($=$ a T-equivariant line bundle on X^{reg}), we put

$$D^T := X^T \frown c_{1,T}(D|_{X^{\mathrm{reg}}}) \in H^{\mathrm{lf},T}_{2n-2}(X,\mathbb{Z}).$$

For a T-equivariant reflexive sheaf D and a T-equivariant complex line bundle L on X, the equivariant intersection is defined as

$$(D^T.L_T^{\cdot n+q-1}) := \int_{X_T^{2q}} D^T \frown c_{1,T}(L)^{\smile n+q-1} \in H^{\mathrm{lf},T}_{-2q}(\mathrm{pt}) \cong H^{2q}_T(\mathrm{pt}).$$

Equivariant intersection 2

Under the Chern–Weil isomorphism (sign convention is important!)

$$H^*_T(\mathrm{pt}, \mathbb{R}) \cong S^* \mathfrak{t}^\vee : c_1(p_i^* \mathcal{O}(-1)) \mapsto (\eta_i^\vee : T \to \mathbb{C}^\times),$$

we can identify $(D^T . L_T^{\cdot n+q-1})$ with a degree q polynomial function on $\mathfrak{t}$.

For $\xi \in \mathfrak{t}$, $(D^T . L_T^{\cdot n+q-1}; \xi) \in \mathbb{R}$ denote the value of the function.
We put

$$(D^T . e^{L_T}; \xi) := \sum_{k=0}^{\infty} \frac{1}{p!} (D^T . L_T^{\cdot p}; \xi) = \sum_{q=0}^{\infty} \frac{1}{(n+q-1)!} (D^T . L_T^{\cdot n+q-1}; \xi).$$

This is compactly absolutely convergent: take a resolution $\beta : \tilde{X} \to X$ and $\tilde{D}_T \in H^2_T(\tilde{X})$ so that $\beta_*(\tilde{X} \frown \tilde{D}_T) = D^T$. Using equiv forms $\Delta + \delta \in \tilde{D}_T, M + \mu \in c_{1,T}(\beta^* L)$, we can compute

$$(D^T . e^{L_T}; \xi) = \int_{\tilde{X}} (\Delta + \delta_\xi) e^{M + \mu_\xi} = \frac{1}{(n-1)!} \int_{\tilde{X}} e^{\mu_\xi} \Delta \wedge M^{n-1} + \frac{1}{n!} \int_{\tilde{X}} \delta_\xi e^{\mu_\xi} M^n.$$

2. Perelman's entropy and μ-cscK metrics

$$\check{\mathcal{W}}^\lambda(\omega_\phi, \varphi)$$

$\inf_{\omega_\phi}$: $\check{\mathcal{W}}^\lambda(\omega_\phi, \varphi) \to \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\varphi)$

$\sup_\varphi$: $\check{\mathcal{W}}^\lambda(\omega_\phi, \varphi) \to \check{\boldsymbol{\mu}}^\lambda(\omega_\phi)$

Minimax conjecture

$$\sup_\varphi \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\varphi) = \inf_{\omega_\phi} \check{\boldsymbol{\mu}}^\lambda(\omega_\phi)$$

Perelman's entropy and μ-cscK metrics

For a Kähler metric $\omega \in L$ and $f \in C^{0,1}(X)$, we put

$$\check{\mathcal{W}}^\lambda(\omega, f) := -\frac{\int_X \left(s(\omega) + |\partial^\sharp f|^2 - \lambda(n+f)\right) e^f \omega^n}{\int_X e^f \omega^n} - \lambda \log \int_X e^f \omega^n / n!.$$

We regard $\check{\mathcal{W}}^\lambda$ as a functional on the tangent bundle

$$T\mathcal{H}(X, L) = \mathcal{H}(X, L) \times C^\infty(X)/\mathbb{R}.$$

Theorem ('21a)

A state (ω, f) is a critical point of $\check{\mathcal{W}}^\lambda$ if and only if $\partial^\sharp f$ is holomorphic and ω is $\check{\mu}^\lambda_\xi$-cscK metric for $\xi = \mathrm{Im}\partial^\sharp f$.
For $\lambda \leq 0$,

$$\check{\boldsymbol{\mu}}^\lambda(\omega) := \sup_f \check{\mathcal{W}}^\lambda(\omega, f)$$

gives a smooth functional on $\mathcal{H}(X, L)$ and its critical points are precisely $\check{\mu}^\lambda$-cscK metric for some ξ.

Critical points of $\check{\boldsymbol{\mu}}^\lambda$ are actually global minimizers (later).

$C^{1,1}$ geodesic rays

A geodesic ray emanating from a Kähler metric $\omega_\phi = \omega + dd^c\phi$ is a $U(1)$-invariant locally bounded $p_X^*\omega$-psh function Φ on $X \times \bar{\Delta}^*$ satisfying

$$\Phi|_{X \times \partial\bar{\Delta}^*} = \phi \quad \text{and} \quad (dd^c_\omega \Phi)^{n+1} = 0.$$

We put $\phi_t := \Phi(\cdot, e^{-t})$ and $\omega_{\phi_t} := \omega + dd^c\phi_t \in \mathrm{PSH}(X, L)$.

For a normal test configuration $(\mathcal{X}, \mathcal{L})$, we can take a smooth function Φ_Ω on $X \times \bar{\Delta}^*$ so that $\omega + dd^c\Phi_\Omega = \Omega|_{X \times \bar{\Delta}^*}$ and $i_\eta d^c\Phi_\Omega = \mu_\eta$ for a smooth equivariant form $\Omega + \mu \in c_{1,U(1)}(\beta^*\bar{\mathcal{L}})$ on $\beta : \tilde{\mathcal{X}} \to \bar{\mathcal{X}}$.

Theorem (Chu–Tossati–Weinkove, (cf. Phong–Sturm))

- For a normal test configuration $(\mathcal{X}, \mathcal{L})$, there exists a unique $C^{1,1}_{\mathrm{loc}}$-geodesic ray Φ such that $\Phi - \Phi_\Omega$ is globally bounded.
- If $\mathcal{X}$ is smooth, $\Phi - \Phi_\Omega$ extends to a $C^{1,1}$-function on $\mathcal{X}$ across the central fibre.

Monotonicity along $C^{1,1}$ geodesic rays

$\check{\mathcal{W}}(\omega_{\phi_t}, -\dot{\phi}_t) = -\int_X (s(\omega_{\phi_t}) + |\partial^\sharp \dot{\phi}_t|^2) e^{-\dot{\phi}_t} \omega_{\phi_t}^n / \int_X e^{-\dot{\phi}_t} \omega_{\phi_t}^n$ is not well-defined for $C^{1,1}$-regular ϕ_t.

Observe for smooth ϕ_t, the antiderivative

$$\int_0^t ds \int_X (s(\omega_{\phi_s}) + |\partial^\sharp \dot{\phi}_s|^2) e^{-\dot{\phi}_s} \omega_{\phi_s}^n$$

can be written as

$$\int_X \frac{d\mu_t}{d\mu_0} \log \frac{d\mu_t}{d\mu_0} d\mu_0 - \int_X (\dot{\phi}_t - \dot{\phi}_0) e^{-\dot{\phi}_t} \omega_{\phi_t}^n + \int_0^t ds \int_X n\mathrm{Ric}(\omega_{\phi_0}) \wedge e^{-\dot{\phi}_s} \omega_{\phi_s}^{n-1}.$$

for $\mu_t = e^{-\dot{\phi}_t} \omega_{\phi_t}^n$. We denote it by $\mathcal{A}_\Phi(t)$.

Theorem ('21a, argument analogous to Berman–Berndtsson)

For any $C^{1,1}_{\mathrm{loc}}$-geodesic ray Φ emanating from a smooth ω_ϕ, $\mathcal{A}_\Phi(t)$ is convex and continuous up to the boundary of $[0, \infty)$ and we have

$$\check{\mathcal{W}}_\flat(\omega_\phi, -\dot{\phi}_0) := -\frac{\frac{d}{dt_+}|_{t=0} \mathcal{A}_\Phi(t)}{\int_X e^{-\dot{\phi}_0} \omega_\phi^n} \leq \check{\mathcal{W}}(\omega_\phi, -\dot{\phi}_0)$$

.

What appears in the limit?: non-archimedean μ-entropy

Theorem ('21a, argument analogous to Z. Sjöström Dyrefelt)

For the geodesic ray Φ subordinate to a *smooth* test configuration $(\mathcal{X}, \mathcal{L})$, we put $\Phi_{;\rho}(x, \tau) := \Phi(x, |\tau|^\rho)$ for $\rho > 0$. Then

$$-\lim_{t \to \infty} \frac{\frac{d}{dt}_+ \mathcal{A}_{\Phi_{;\rho}}(t)}{\int_X e^{-\dot{\phi}_{;\rho,t}} \omega^n_{\phi_{;\rho,t}}} = 2\pi \frac{(K_X . e^L) - \frac{\rho}{\pi}(K^{\log, \mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} . e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}} ; \frac{\rho}{\pi})}{(e^L) - \frac{\rho}{\pi}(e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}} ; \frac{\rho}{\pi})}.$$

Here $K^{\log, \mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} := (K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}} + \mathcal{X}_0^{\mathrm{red}, \mathbb{C}^\times}) - \varpi^*(K^{\mathbb{C}^\times}_{\mathbb{P}^1} + 0^{\mathbb{C}^\times})$.

We put $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) := \check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) + \lambda \check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \rho)$

$$\check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) := 2\pi \frac{(K_X . e^L) - \rho(K^{\log, \mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} . e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}} ; \rho)}{(e^L) - \rho(e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}} ; \rho)}$$

$$\check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \rho) := \frac{(L . e^L) - \rho(\bar{\mathcal{L}}_{\mathbb{C}^\times} . e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}} ; \rho)}{(e^L) - \rho(e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}} ; \rho)} - \log((e^L) - \rho(e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}} ; \rho))$$

What appears in the limit?: non-archimedean μ-entropy

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For the geodesic ray Φ subordinate to a *smooth* test configuration $(\mathcal{X}, \mathcal{L})$, we put $\Phi_{;\rho}(x, \tau) := \Phi(x, |\tau|^\rho)$ for $\rho > 0$. Then

$$- \lim_{t \to \infty} \frac{\frac{d}{dt_+} \mathcal{A}_{\Phi_{;\rho}}(t)}{\int_X e^{-\dot{\phi}_{;\rho,t}} \omega^n_{\phi_{;\rho,t}}} = 2\pi \frac{(K_X . e^L) - \frac{\rho}{\pi} (K^{\log, \mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} . e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}} ; \frac{\rho}{\pi})}{(e^L) - \frac{\rho}{\pi} (e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}} ; \frac{\rho}{\pi})}.$$

Here $K^{\log, \mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} := (K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}} + \mathcal{X}_0^{\mathrm{red}, \mathbb{C}^\times}) - \varpi^* (K^{\mathbb{C}^\times}_{\mathbb{P}^1} + 0^{\mathbb{C}^\times})$.

We put $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) := \check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) + \lambda \check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \rho)$

$$\check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) := 2\pi \frac{(\kappa^{\mathbb{C}^\times}_{\mathcal{X}_0} . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0} ; \rho) - \rho (\mathcal{X}_0^{\mathrm{red}, \mathbb{C}^\times} - \mathcal{X}_0^{\mathbb{C}^\times} . e^{\mathcal{L}_{\mathbb{C}^\times}} ; \rho)}{\int_{\mathbb{R}} e^{-\rho t} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}}$$

$$\check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \rho) := \frac{\int_{\mathbb{R}} (n - \rho t) e^{-\rho t} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}}{\int_{\mathbb{R}} e^{-\rho t} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}} - \log \int_{\mathbb{R}} e^{-\rho t} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}$$

Summary on W-entropy

Let $\mathcal{H}_{\mathrm{NA}}(X, L)$ denote the set of normal test configurations and φ denote its element. Later, we will identify φ with a function on the Berkovich space X^{NA}. Using the geodesic ray $\Phi = \{\phi_t\}$ emanating from ω_ϕ subordinate to φ, we put

$$\check{\mathcal{W}}^\lambda(\omega_\phi, \varphi) := \check{\mathcal{W}}^\lambda_\flat(\omega_\phi, -\dot{\phi}_0)$$

Summary

To sum up, we have

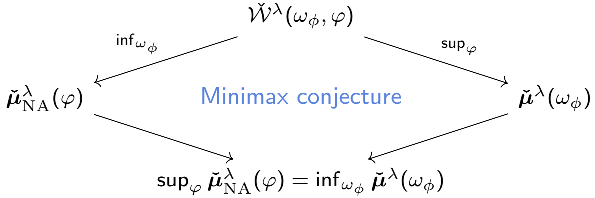
$$\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\varphi) \le \inf_{\omega_\phi \in \mathcal{H}(X,L)} \check{\mathcal{W}}^\lambda(\omega_\phi, \varphi) \le \sup_{\varphi \in \mathcal{H}_{\mathrm{NA}}(X,L)} \check{\mathcal{W}}^\lambda(\omega_\phi, \varphi) \le \check{\boldsymbol{\mu}}^\lambda(\omega_\phi).$$

Proposition ('21a)

If (X, L) admits a $\check{\mu}^\lambda_\xi$-cscK metric ω for $\lambda \le 0$, then

$$\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\xi) = \sup_{\varphi \in \mathcal{H}_{\mathrm{NA}}(X,L)} \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\varphi) = \inf_{\omega_\phi \in \mathcal{H}(X,L)} \check{\boldsymbol{\mu}}^\lambda(\omega_\phi) = \check{\boldsymbol{\mu}}^\lambda(\omega).$$

A conjectural picture: minimax picture



cf. Sion's minimax theorem

Let W be a convex subset of a linear topological space and C be a compact convex subset of a linear topological space. If $f : W \times C \to \mathbb{R}$ is a function satisfying

- $f(w, \cdot) : C \to \mathbb{R}$ is usc and quasi-concave: $f(w, \cdot)^{-1}((-\infty, a))$ is convex,
- $f(\cdot, c) : W \to \mathbb{R}$ is lsc and quasi-convex: $f(\cdot, c)^{-1}((a, \infty))$ is convex,

then we have

$$\sup_{c \in C} \inf_{w \in W} f(w, c) = \inf_{w \in W} \sup_{c \in C} f(w, c).$$

Non-archimedean μ-entropy and μK-semistability

Theorem ('21b, essentially observed in '20)

If $\varphi \in \mathcal{H}_{\mathrm{NA}}(X, L)$ maximizes $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$, then the central fibre of φ is μ^λK-semistable with respect to the vector ξ generated by the $\mathbb{C}^\times$-action.

We obtain the following criterion for μK-stability

Proposition ('19 + '21b)

Suppose for every test configuration $(\mathcal{X}, \mathcal{L})$ there exists $\xi \in N_{\mathbb{Q}}$ such that $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \leq \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\xi)$. Then (X, L) is μ^λK-semistable for some ξ.

and a new proof for the following

Corollary ('21b)

If (X, L) admits a $\check{\mu}^\lambda_\xi$-cscK metric for $\lambda \leq 0$, then it is $\check{\mu}^\lambda_\xi$K-semistable.

Toric case

Let (X, L) be a normal toric variety and P be the moment polytope. For a toric test configuration $(\mathcal{X}, \mathcal{L})$ and $c \gg 0$, $\bar{\mathcal{L}}_c = \bar{\mathcal{L}} + c\mathcal{X}_0$ for $\mathcal{L}_c = \mathcal{L} + c.\mathbb{C}_{-1}$ is ample. The associated polytope Q_c is of the form $Q_c = \{(\mu, t) \in P \times \mathbb{R} \mid 0 \leq t \leq -q(\mu) + c\}$ for a convex function q on P.

Proposition ('21b, essentially observed in '20)

$$\check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) = -2\pi \frac{\int_{\partial P} e^{\rho q} d\sigma}{\int_P e^{\rho q} d\mu},$$

$$\check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \rho) = \frac{\int_P (n + \rho q) e^{\rho q} d\mu}{\int_P e^{\rho q} d\mu} - \log \int_P e^{\rho q} d\mu.$$

Eg. When $n \leq 2$, there exists an lsc convex function q on P maximizing $\check{\boldsymbol{\mu}}_{\mathrm{NA}}$ with $\int_P e^q d\mu < \infty$.

3. Moment measure and non-archimedean μ-entropy

$$\begin{array}{ccc} \mathcal{H}_{\mathrm{NA}}(X, L) & \xrightarrow{\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}} & [-\infty, \infty) \\ \downarrow & \nearrow_{\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}} & \\ \mathcal{E}^{\exp}_{\mathrm{NA}}(X, L) & & \end{array}$$

$$-\frac{\int_{X^{\mathrm{NA}}} (2\pi A_X + \lambda\varphi) \int e^{-t} \mathcal{D}_\varphi + E^{2\pi K_X + \lambda L}_{\exp}(\varphi)}{\iint_{X^{\mathrm{NA}}} e^{-t} \mathcal{D}_\varphi} - \lambda \log \iint_{X^{\mathrm{NA}}} e^{-t} \mathcal{D}_\varphi$$

Test configuration, revisit

Let (X, L) be a T-equivariant polarized variety. A T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L) is a $\mathbb{C}^\times \times T$–equivariant flat family of schemes over $\mathbb{A}^1 = \mathbb{C}$ endowed with

- a relatively ample $\mathbb{C}^\times \times T$-equivariant $\mathbb{Q}$-line bundle $\mathcal{L}$ and
- a specific isomorphism $(X, L) \cong (\mathcal{X}_1, \mathcal{L}|_{\mathcal{X}_1})$ of the fibre over $1 \in \mathbb{A}^1$.

We have a natrual compactification $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ over $\mathbb{P}^1$.

We assign the following filtration $\mathcal{F}_{(\mathcal{X},\mathcal{L})}$ on $R = \bigoplus_m R_m = \bigoplus_m H^0(X, L^{\otimes m})$:

$$\mathcal{F}^\lambda_{(\mathcal{X},\mathcal{L})} R_m := \{s \in R_m \mid \varpi^{-\lceil \lambda \rceil} \bar{s} \text{ extends to a section of } \mathcal{L}^{\otimes m}\}.$$

For a *normal* test configuration $(\mathcal{X}, \mathcal{L})$ and an irreducible component $E \subset \mathcal{X}_0$, we assign the following valuation v_E on $\mathbb{C}(X)$:

$$v_E(f) := \frac{\mathrm{ord}_E(f \circ p_X)}{\mathrm{ord}_E \mathcal{X}_0}$$

Global pluripotential theory over trivially valued fields

The Berkovich space X^{NA} is a compactification of $\mathrm{Val}(X)$. It consists of semi-valuations v: $v \in \mathrm{Val}(Y)$ for some irreducible variety $Y \subset X$.

We consider the following function $\varphi_{(\mathcal{X},\mathcal{L})}$ on X^{NA} for a test configuration $(\mathcal{X},\mathcal{L})$:

$$\varphi_{(\mathcal{X},\mathcal{L})}(v) := \inf\{\sigma \in \mathbb{R} \mid \mathcal{F}_{(\mathcal{X},\mathcal{L})} \subset \mathcal{F}_v[\sigma]\},$$

where $\mathcal{F}_v[\sigma]^\lambda R_m := \{s \in R_m \mid v(s) + m\sigma \geq \lambda\}$.
We have $\varphi_{(\mathcal{X},\mathcal{L})} = \varphi_{(\mathcal{X}',\mathcal{L}')}$ iff the normalization coincides:

$$\mathcal{H}_{\mathrm{NA}}(X,L) = \{\varphi_{(\mathcal{X},\mathcal{L})}\}.$$

Definition (Boucksom–Jonsson)

A non-archimedean psh metric on (X,L) is a function on X^{NA} given as the limit of a decreasing net $\{\varphi_{(\mathcal{X}_i,\mathcal{L}_i)}\}$. We denote the set of NA psh metrics by $\mathrm{PSH}_{\mathrm{NA}}(X,L)$.

Non-archimedean Monge–Ampère measure

For $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(X, L)$, we put

$$E(\varphi) := \inf \left\{ \frac{(\bar{\mathcal{L}}^{\cdot n+1})}{(n+1)!} = \int_{\mathbb{R}} t \mathrm{DH}_{(\mathcal{X},\mathcal{L})} \;\middle|\; \varphi \leq \varphi_{(\mathcal{X},\mathcal{L})} \in \mathcal{H}_{\mathrm{NA}}(X, L) \right\}$$

and

$$\mathcal{E}^1_{\mathrm{NA}}(X, L) := \{\varphi \in \mathrm{PSH}_{\mathrm{NA}}(X, L) \mid E(\varphi) > -\infty\}.$$

Theorem (Boucksom–Jonsson)

For $\varphi \in \mathcal{E}^1_{\mathrm{NA}}(X, L)$, we can assign a measure $\mathrm{MA}(\varphi)$ on X^{NA} with a continuity property along decreasing nets $\varphi_i \searrow \varphi$. For a normal test configuration $\varphi \equiv (\mathcal{X}, \mathcal{L}) \in \mathcal{H}_{\mathrm{NA}}(X, L)$, the measure is explicitly given by:

$$\mathrm{MA}(\varphi) := \frac{1}{n!} \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E \mathcal{X}_0 \cdot (E.\mathcal{L}^{\cdot n}).\delta_{v_E}.$$

We use a compact Hausdorff topology on X^{NA} $\rightsquigarrow$ we can apply Dini's lemma and Riesz–Markov–Kakutani theorem.

Log discrepancy

For a prime divisor $E \subset \tilde{X} \to X$ over X, we put

$$A_X(c.\mathrm{ord}_E) = c(1 + \mathrm{ord}_E(K_{\tilde{X}/X}))$$

The log discrepancy A_X extends to a lsc functional on X^{NA}.

Recall how the Mabuchi invariant M^{NA} ($\approx$ Donaldson–Futaki invariant) is extended to $\mathcal{E}^1_{\mathrm{NA}}(X, L)$:

$$M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{1}{n!}(K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1}.\bar{\mathcal{L}}^{\cdot n}) - \frac{n(K_X.L^{\cdot n-1})}{(L^{\cdot n})}\frac{(\bar{\mathcal{L}}^{\cdot n+1})}{(n+1)!}.$$

Using $K^{\log}_{\bar{\mathcal{X}}/X_{\mathbb{P}^1}} = \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E \mathcal{X}_0 \cdot A_X(v_E).E$,

$$\begin{aligned} M^{\mathrm{NA}}(\varphi) &= \frac{1}{n!}(K^{\log}_{\bar{\mathcal{X}}/X_{\mathbb{P}^1}}.\bar{\mathcal{L}}^{\cdot n}) + \frac{1}{n!}(K_{X_{\mathbb{P}^1}/\mathbb{P}^1}.\bar{\mathcal{L}}^{\cdot n}) - \frac{n(K_X.L^{\cdot n-1})}{(L^{\cdot n})}E(\varphi) \\ &= \int_{X^{\mathrm{NA}}} A_X \mathrm{MA}(\varphi) + E^{K_X}(\varphi) - \frac{n(K_X.L^{\cdot n-1})}{(L^{\cdot n})}E(\varphi). \end{aligned}$$

Towards non-archimedean formalism: moment measure

For a test configuration $\varphi \equiv (\mathcal{X}, \mathcal{L})$, we consider the following measure $\mathcal{D}_\varphi$ on $\mathbb{R} \times X^{\mathrm{NA}}$:

$$\mathcal{D}_\varphi := \sum_E \mathrm{ord}_E \mathcal{X}_0 . \mathrm{DH}_{E, \mathcal{L}|_E} \otimes \delta_{v_E}.$$

Then we have

$$\int_{X^{\mathrm{NA}}} \int_{\mathbb{R}} \chi(-\rho t) \mathcal{D}_\varphi = (\mathcal{X}_0^{\mathbb{C}^\times} . \chi(\bar{\mathcal{L}}_{\mathbb{C}^\times}); \rho),$$

$$\int_{X^{\mathrm{NA}}} A_X \int_{\mathbb{R}} e^{-\rho t} \mathcal{D}_\varphi = (\rho K^{\log, \mathbb{C}^\times}_{\bar{\mathcal{X}}/X_{\mathbb{P}^1}} . e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \rho).$$

For a $\mathbb{C}^\times$ action on (X, L) and a $U(1)$-invariant metric $\omega \in L$, we can associate a unique moment map $\mu : X \to \mathbb{R}$ normalized by $[\omega + \mu] = L_{\mathbb{C}^\times}$. The measure $\mathcal{D} = (\mathrm{id}_X \times \mu)_* \omega^n$ on $\mathbb{R} \times X$ determines μ and ω. For $\chi \in C^\infty(\mathbb{R})$, we have

$$\int_X \psi \int_{\mathbb{R}} \chi(t) \mathcal{D} = \int_X \psi \cdot \chi \circ \mu \, \omega^n.$$

Main results 1: moment measure

Theorem ('21b)

For general $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(X, L)$, we can assign a measure DH_φ on $\mathbb{R}$ with a continuity property along decreasing nets $\varphi_i \searrow \varphi$. For $\varphi \equiv (\mathcal{X}, \mathcal{L}) \in \mathcal{H}_{\mathrm{NA}}(X, L)$, we have

$$\mathrm{DH}_\varphi = \mathrm{DH}_{(\mathcal{X},\mathcal{L})} = \lim_{m \to \infty} \frac{1}{m^n} \sum_{\lambda \in \mathbb{Z}} \dim(\mathcal{F}^\lambda R_m / \mathcal{F}^{\lambda+} R_m).\delta_{\lambda/m}.$$

For $\varphi \in \mathcal{E}^1_{\mathrm{NA}}(X, L)$ and a Borel measurable function χ on $\mathbb{R}$ with $\int_{\mathbb{R}} |\chi| \mathrm{DH}_\varphi < \infty$, we can assign a measure $\int \chi \mathcal{D}_\varphi$ on X^{NA} with a continuity property along decreasing nets. For a normal test configuration $\varphi \equiv (\mathcal{X}, \mathcal{L}) \in \mathcal{H}_{\mathrm{NA}}(X, L)$, the measure is explicitly given by:

$$\int \chi \mathcal{D}_\varphi = \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E \mathcal{X}_0 \cdot \int_{\mathbb{R}} \chi \mathrm{DH}_{E, \mathcal{L}|_E} . \delta_{v_E}.$$

Construction of moment measure

For $\varphi \in \mathcal{E}^1_{\mathrm{NA}}(X, L)$ and $\tau \in \mathbb{R}$, $\varphi \wedge \tau := \sup\{\varphi' \in \mathrm{PSH}_{\mathrm{NA}}(X, L) \mid \varphi' \leq \varphi, \tau\}$ gives a non-archimedean psh metric in $\mathcal{E}^1_{\mathrm{NA}}(X, L)$.

For $\varphi \equiv (\mathcal{X}, \mathcal{L}) \in \mathcal{H}_{\mathrm{NA}}(X, L)$, we have

$$\mathrm{MA}(\varphi \wedge \tau) = \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E \mathcal{X}_0 \int_{(-\infty, \tau)} \mathrm{DH}_{E, \mathcal{L}|_E}.\delta_{v_E} + \int_{[\tau, \infty)} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}.\delta_{v_{\mathrm{triv}}}.$$

1. We put $\int 1_{[\tau', \tau)} \mathcal{D}_\varphi := \mathrm{MA}(\varphi \wedge \tau) - \mathrm{MA}(\varphi \wedge \tau') + \int_{[\tau', \tau)} \mathrm{DH}_\varphi.\delta_{v_{\mathrm{triv}}}$.
2. For a non-negative $g \in C^0(X^{\mathrm{NA}})$ and a Borel measurable $B \subset \mathbb{R}$, we put
$$\nu_{\varphi, g}(B) := \inf\{\sum_{i=1}^\infty \int_{X^{\mathrm{NA}}} g \int 1_{[\tau_i', \tau_i)} \mathcal{D}_\varphi \mid B \subset \bigcup_{i=1}^\infty [\tau_i', \tau_i)\}.$$
3. $I_{\varphi, \chi}(g) := \int_{\mathbb{R}} \chi \nu_{\varphi, g}$ gives a bounded linear functional on $C^0(X^{\mathrm{NA}})$, then by representation theorem, we get a measure $\int \chi \mathcal{D}_\varphi$ satisfying
$$\int_{X^{\mathrm{NA}}} g \int \chi \mathcal{D}_\varphi = I_{\varphi, \chi}(g).$$

Main results 2: continuous extension to $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$

For $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(X, L)$ and $\rho > 0$, we put

$$E_{\exp}(\varphi_{;\rho}) := \inf \Big\{ -\int_{\mathbb{R}} e^{-\rho t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})} \ \Big| \ \varphi \leq \varphi_{(\mathcal{X},\mathcal{L})} \in \mathcal{H}_{\mathrm{NA}}(X, L) \Big\}$$

and

$$\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L) := \{\varphi \in \mathrm{PSH}(X, L) \mid E_{\exp}(\varphi_{;\rho}) > -\infty \text{ for } \forall \rho > 0\}.$$

Theorem ('21b)

There is a metric $d_{\exp}$ on $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$ for which $\varphi \mapsto \int e^{-t} \mathcal{D}_\varphi$ is continuous. Moreover, there is a continuous extension of the following to $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$:

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\varphi) + \frac{\int_{X^{\mathrm{NA}}} A_X \int e^{-t} \mathcal{D}_\varphi}{\iint_{X^{\mathrm{NA}}} e^{-t} \mathcal{D}_\varphi}.$$

Suppose X is smooth (or the continuity of envelopes holds for (X, L)). Then the metric space $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$ is complete.

Continuity: tomographic expression

For $\varphi \in \mathcal{E}^1_{\mathrm{NA}}(X, L)$ and $\tau \in \mathbb{R}$,
$\varphi \wedge \tau = \sup\{\varphi' \in \mathrm{PSH}_{\mathrm{NA}}(X, L) \mid \varphi' \leq \varphi, \tau\}$ exists in $\mathcal{E}^1_{\mathrm{NA}}(X, L)$ (without assuming the continuity of envelopes).

Proposition ('21b)

We have

$$\int_{X^{\mathrm{NA}}} \psi \int e^{-t} \mathcal{D}_\varphi = \int_{\mathbb{R}} d\tau \; e^{-\tau} \int_{X^{\mathrm{NA}}} (\psi - \psi(v_{\mathrm{triv}})) \mathrm{MA}(\varphi \wedge \tau) + \psi(v_{\mathrm{triv}}) \int_{\mathbb{R}} e^{-\tau} \mathrm{DH}_\varphi(\tau)$$

for $\psi \in \mathcal{E}^1_{\mathrm{NA}}(X, L)$ and $\varphi \in \mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$. Moreover, we have

$$\left| e^{-\tau} \int_{X^{\mathrm{NA}}} (\psi - \psi(v_{\mathrm{triv}})) \mathrm{MA}(\varphi \wedge \tau) \right| \leq C_\varepsilon e^{-\varepsilon|\tau|}$$

for C_ε depending boundedly on $n, (L^{\cdot n}), d_1(\varphi, 0), d_1(\psi, 0), E_{\exp}(\varphi_{;2+2\varepsilon})$.

The non-archimedean μ-entropy on $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$

Corollary ('21b)

The non-archimedean μ-entropy $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\varphi) =$

$$-\frac{\int_{X^{\mathrm{NA}}} (2\pi A_X + \lambda\varphi) \int e^{-t}\mathcal{D}_\varphi + E^{2\pi K_X + \lambda L}_{\exp}(\varphi)}{\iint_{X^{\mathrm{NA}}} e^{-t}\mathcal{D}_\varphi} - \lambda \log \iint_{X^{\mathrm{NA}}} e^{-t}\mathcal{D}_\varphi$$

is well-defined and upper semi-continuous on $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$.

Conjecture

The subset

$$\left\{ \varphi \in \mathcal{E}^{\exp}_{\mathrm{NA}}(X, L) \;\middle|\; \iint_{X^{\mathrm{NA}}} e^{-t}\mathcal{D}_\varphi = 1,\ \int_{X^{\mathrm{NA}}} A_X \int e^{-t}\mathcal{D}_\varphi + E^{K_X}_{\exp}(\varphi) \leq C \right\}$$

is weakly compact and $E_{\exp}(\varphi_{;\rho}) = -\iint_{X^{\mathrm{NA}}} e^{-\rho t}\mathcal{D}_\varphi$ is bounded for any $\rho > 0$.

If the conjecture holds, then there exists a maximizer of $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$ for $\lambda \leq 0$. We call it a non-archimedean μ-cscK metric.

Thank you for listening!