Moment polytopes on Sasaki manifolds and volume minimization

Akito Futaki
Tsinghua University, Yau Mathematical Sciences Center

November 8-11, 2022
Kanazawa University (online)

Let S be a compact smooth manifold of $\dim S = 2m + 1$.

A Sasakian structure on S consists of

- a contact distribution D, i.e. a $2m$ dimensional distribution i.e.
TS/D is trivial and oriented,
the Levi form $L_D : D \times D \to TS/D$ defined by
$L_D(X,Y) = -\eta_D([X,Y])$ is nodegenerate
where $\eta_D : TS \to TS/D$ is the projection,

- pseudo-convex CR-structure J on D i.e.
$L_D(X, JY)$ is positive definite Hermitian form.

- Reeb vector field ξ i.e.
a CR vector field ($\Leftrightarrow$ Lie derivative L_ξ prserves $C^\infty(D)$ and J),
$\eta_D(\xi)$ is a positive section of $C^\infty(TS/D)$.

$$TS = D \oplus \mathbb{R}\xi$$

Example : $S =$ the unit circle bundle of a positive line bundle $L \to M$

The flow $\{\mathrm{Exp}(t\xi)\}_{t\in\mathbb{R}}$ is called the Reeb flow.

The pseudo-convex CR-structure determines Kähler structures on the local orbit spaces of the Reeb flow.

Differential forms on S obtained by pulling back from those local orbit spaces are called *basic forms*.

Naturally ∂ and $\overline{\partial}$ operators can be considered to operate on basic forms, which we denote by ∂_B and $\overline{\partial}_B$, and we obtain Dolbeault theory, Hodge theory and Chern-Weil theory for basic forms.

Suppose that the basic first Chern class $c_1^B(S)$ is positive, (i.e. represented by a real closed positive $(1,1)$-basic form) or zero or negative.
Each case we take $\epsilon = 1, 0, -1$.

Suppose that we are given a decomposition

$$2\pi c_1^B(S) = \epsilon(\gamma_1 + \cdots + \gamma_k)$$

for some basic Kähler classes γ_α.

Basic Kähler metrics $\omega_\alpha \in \gamma_\alpha$ are called *transverse coupled Kähler-Einstein metrics* if

$$\rho^T(\omega_1) = \cdots = \rho^T(\omega_k) = \epsilon \sum_{\beta=1}^{k} \omega_\beta \qquad (1)$$

where

$$\rho^T(\omega_\alpha) = -i\partial_B\overline{\partial}_B \log \omega_\alpha^m$$

is the transverse Ricci form of ω_α.

In the case of $\epsilon = 0$ and -1, by Yau's theorem a solution always exists.

Defn : A Sasaki manifold of $\dim 2m + 1$ is said to be *toric* if S admits an effective T^{m+1}-action preserving (D, J) and $\xi \in \mathfrak{t}$.

In the case of $k = 1$ and $\epsilon = 1$, F-Ono-Wang proved the following: Let S be a toric Sasaki manifold with $c_1^B(S) > 0$ and $c_1(D) = 0$. Then there exists another Sasakian structure (D', J', ξ') for which a transverse Kähler-Einstein metrics exists. This induces a Sasaki-Einstein metric on S.

Essential idea is the volume minimization of Martelli-Sparks-Yau.

Question : Does a similar idea work for the coupled transverse KE metrics ?

Kähler manifolds

$\subset$ Riemannian manifolds $\cap$ complex manifolds $\cap$ symplectic manifolds

Sasaki manifolds

$\subset$ Riemannian manifolds $\cap$ CR manifolds $\cap$ contact manifolds

Another aspect:

Kähler cone $\mathbb{R} \times S, dr^2 + r^2 g_S$ (affine algebraic variety)

$\supset \{r = 1\} = S$

$\to$ Kähler local orbit spaces of the Reeb flow

"Kähler sandwich" (Boyer-Galicki)

More on the definitions of Sasaki manifolds.

As a Riemannian manifold

S has a natural Riemannian metric g defined by

$$g(\xi, \xi) = 1, \qquad\qquad g(\xi, D) = 0,$$

and

$$g_D = \frac{1}{2} d\eta_\xi(\cdot \ J\cdot)$$

where $\eta_\xi = \eta_D(\xi)^{-1}\eta_D$ contact 1-form.

The Riemannian manfold (S, g) is often called a Sasaki manifold.

The normalization of $g(\xi, \xi) = 1$ is the standard choice.

The associated *transverse Kähler form* ω^T is given by

$$\omega^T = \frac{1}{2} d\eta_\xi.$$

If (S, g) is an Einstein manifold, called a *Sasaki-Einstein (SE for short) manifold*, then the Ricci curvature Ricci satisfies

$$\rho^T(\omega^T) = (m+1)\,\omega^T.$$

Naturally, by the Chern-Weil theory,

$$2\pi c_1^B(S) = [\rho^T(\omega_\alpha)]_B$$

where $[\cdot]_B$ denotes a basic cohomology class.

Hence

$$2\pi c_1^B(S) = (m+1)[\frac{1}{2} d\eta_\xi]_B$$

is an obvious necessary condition for the $\exists$ of a SE metric.

Lemma
Condition $c_1^B(S) > 0$ and $c_1(D) = 0$ is equivalent to

$$2\pi c_1^B(S) = (m+1)[\frac{1}{2}d\eta_\xi]_B$$

after suitable modification (called D-homothetic transformation) of the Sasakian structure.

Transverse moment map w.r.t. $c_1^B(S)$
versus
contatct moment map w.r.t. $[d\eta_\xi]_B$ (or η_ξ)

More on the definitions of Sasaki manifolds.

As a link of a Kähler cone

Fact : Consider the Riemannian cone (V, g_V) of (S, g), where V the product manifold $V = \mathbb{R}_+ \times S$ and the metric g_V is the warped product metric

$$g_V = dr^2 + r^2 g$$

with r the standard coordinate of $\mathbb{R}_+$. Then (V, g_V) is a Kähler.

Conversely, for a Kähler cone manifold V, g_V, the link $\{r = 1\} \cong (S, g)$ is a Sasaki manifold in the following way:

$$\eta = d^c \log r|_{S=\{r=1\}}$$

is a contact form, and $D = \ker \eta$ is a contact distribution, and $\xi := Jr\frac{\partial}{\partial r}|_{r=1}$ is the Reeb vector field.

Summary:

Cone (V, g_V) is Kähler.
$\Leftrightarrow$
$(S, g) \cong \{r = 1\}$ is Sasakian.
$\Leftrightarrow$
The local orbit spaces of the Reeb flow are Kähler.
(Boyer-Galicki called this "Kähler sandwich")

Fact Cone (V, g_V) is Calabi-Yau (Ricci-flat Kähler).
$\Leftrightarrow$
$(S, g) \cong \{r = 1\}$ is Sasaki-Einstein.
$\Leftrightarrow$
The local orbit spaces of the Reeb flow are Kähler-Einstein with

$$\rho^T(\omega^T) = (m + 1)\,\omega^T.$$

Transverse Kähler geometry with ξ fixed:

Let φ be a basic function.
If we change

$$\omega^T := \frac{1}{2} d\eta = \frac{1}{2} dd^c \log r|_{r=1} \in \frac{1}{m+1} c_1^B(S)$$

to

$$\omega(\varphi) = \omega^T + i\partial_B \overline{\partial}_B \varphi = \frac{1}{2} dd^c \log(r e^{2\varphi})|_{\{r=1\}}$$

this corresponds to the change η into $\eta + d^c\varphi$.

So $D = \ker \eta$ changes into $D' = \ker(\eta + d^c\varphi)$
while the Reeb vector field ξ unchanged.

But just as in ordinary Kähler geometry;
Claim : The moment map image depends only on basic Kähler class.
Independent of the CR structure.

Suppose that a real compact torus T acts effectively on S preserving (D, J, ξ) and that the Lie algebra $\mathfrak{t}$ of T contains ξ.

Let ω be a T-invariant basic Kähler form in $\frac{1}{m+1}c_1^B(S)$.

Let F be a T-invariant basic smooth function on S such that

$$\rho^T(\omega) = (m+1)\omega + i\partial_B\overline{\partial}_B F.$$

Then since $c_1^B(S) > 0$, for any $X \in \mathfrak{t}$ there exists a smooth basic function v such that

$$i(X)\omega = -dv.$$

Then as in Fano manifolds

$$\begin{aligned} \mathfrak{t}/\mathbb{R}\xi &\cong \{v \mid \Delta_B v + v^i F_i + (m+1)v = 0\} \\ &\cong \{v \mid i(X)\omega = -dv \quad \text{for some } X \in \mathfrak{t}, \ \int_S v\, e^F \omega^m \wedge \eta_\xi = 0\} \end{aligned}$$

where Δ_B denotes the $\overline{\partial}_B$-Laplacian.

Define Fut : $\mathfrak{t} \to \mathbb{R}$ by

$$\begin{aligned}\mathrm{Fut}(X) &= \int_S v^i F_i \; \omega^m \wedge \eta_\xi \\ &= \int_S \frac{1}{2}(JX)F \, \omega^m \wedge \eta_\xi \\ &= -(m+1) \int_S v \, \omega^m \wedge \eta_\xi .\end{aligned}$$

Then as for Fano manifolds, Fut is independent of choice of ω in $\frac{1}{m+1} c_1^B(S)$, and the non-vanishing of Fut obstructs the existence of a transverse Kähler-Einstein metric in $\frac{1}{m+1} c_1^B(S)$.

This invariant can be expressed in terms of the transverse moment map $\mu^T : S \to (\mathfrak{t}/\mathbb{R}\xi)^*$

$$\langle \mu^T(x), X \rangle = v(x).$$

The image of μ^T is a compact convex polytope, which we write $\frac{1}{m+1}\mathcal{P}_{-K_S}$,

and this polytope is unchaged even if the Kähler form ω is changed in the cohomology class $\frac{1}{m+1}c_1^B(S)$.

Recall the lemma:
Condition $c_1^B(S) > 0$ and $c_1(D) = 0$ is equivalent to

$$2\pi c_1^B(S) = (m+1)[\frac{1}{2}d\eta_\xi]_B.$$

The transverse moment map can be compared to the <u>contact moment map</u> $\mu^{con} : V \to \mathfrak{t}^*$ defined on the Kähler cone V

$$\langle \mu^{con}(x), X \rangle = (r^2 \eta_\xi(X))(x).$$

with

$$v = \frac{m+1}{2}\eta_\xi(X) + c$$

where c is determined by the normalization of v.

Identifying S with $\{r=1\}$ we have the moment map of S by restricting μ^{con} to $\{r=1\}$.

The image of S is

$$\mathcal{P}_\xi := \text{Image}(\mu^{con}) \cap \{p \in \mathfrak{t}^* \,|\, \langle p, \xi \rangle = 1\}$$

since $\eta_\xi(\xi) = 1$.

Since the Hamiltonian functions for the basis of $\mathfrak{t}/\mathbb{R}\xi$ determine affine coordinates on the images of μ^T and μ^{con}, the map

$$\Phi := \mu^T \circ (\mu^{con})^{-1}|_{\mathcal{P}_\xi} : \mathcal{P}_\xi \to \frac{1}{m+1}\mathcal{P}_{-K_S} \qquad (2)$$

is an affine map in terms of those affine coordinates.

$\frac{1}{m+1}\mathcal{P}_{-K_S}$ is in $(\mathfrak{t}/\mathbb{R}\xi)^*$ which is a vector space and contains the origin 0 but that $\mathcal{P}_\xi$ is in a hyperplane in the cone $\mathcal{C} \subset \mathfrak{t}^*$.

Put $\mathbf{o} := \Phi^{-1}(0)$, and call $\mathbf{o}$ the *origin* in the image $\mathcal{P}_\xi \subset \{p \in \mathfrak{t}^* \,|\, \langle p, \xi \rangle = 1\}$ of the contact moment map.

We regard the hyperplane $\{p \in \mathfrak{t}^* \,|\, \langle p, \xi \rangle = 1\}$ as a vector space by choosing the origin $\mathbf{o}$ to be zero,

Suppose also that we are given a decomposition

$$2\pi c_1^B(S) = \gamma_1 + \cdots + \gamma_k.$$

Proposition

(1) There is a Minkowski sum decomposition

$$\mathcal{P}_\xi = \mathcal{P}_{\xi,1} + \cdots + \mathcal{P}_{\xi,k}$$

into the sum of convex polytopes $\mathcal{P}_{\xi,\alpha} \subset \mathcal{P}_\xi$, such that if there are transverse coupled Kähler-Einstein metrics then the sum of the barycenters of $\mathcal{P}_{\xi,\alpha}$ lies at the origin $\mathbf{o}$.

(2) The Minkowski sum decomposition in (1) is unique up to translations of $\mathcal{P}_{\xi,\alpha}$ to $\mathcal{P}_{\xi,\alpha} + c_\alpha$ with $c_\alpha \in \mathfrak{t}^*$ such that $\sum_{\alpha=1}^{k} c_\alpha = \mathbf{o}$.

(3) The Minkoswski sum decomposition of $\mathcal{P}_\xi$ in (1) determines a Minkowski sum decomposition of the contact moment cone $\mathcal{C}_\xi$

$$\mathcal{C}_\xi = \mathcal{C}_{\xi,1} + \cdots + \mathcal{C}_{\xi,k}$$

into the sum of cones $\mathcal{C}_{\xi,\alpha} \subset \mathfrak{t}^*$ in such a way that the intersection of $\mathcal{C}_{\xi,\alpha}$ with $\mathcal{P}_\xi$ is $\mathcal{P}_{\xi,\alpha}$.

A Sasaki manifold S of dimension $2m+1$ is *toric* if its Kähler cone $C(S)$ is toric.

For a compact toric Sasaki manifold S we have the following equivalent conditions, c.f. Martelli-Sparks-Yau, Cho-F-Ono:

• $c_1^B(S) > 0$ and $c_1(D) = 0$.
• There is a rational vector $\gamma \in \mathfrak{t}^*$ such that

$$\langle \gamma, \xi \rangle = -m - 1 \quad \text{and} \quad \ell_a(\gamma) = -1 \text{ for } a = 1, \cdots, d.$$

where the moment cone of $C(S)$ is

$$\mathcal{C} = \{p \in \mathfrak{t}^* \backslash \{o\} \,|\, \langle p, \ell_a \rangle \geq 0,\ a = 1, \cdots, d\}$$

$2\pi\ell_1, \cdots, 2\pi\ell_d$ are primitive elements of the kernel Λ of $\exp : \mathfrak{t} \to T$.

• The power of the canonical line bundle $K_{C(S)}^{\otimes \ell}$ of the cone $C(S)$ is a trivial line bundle for some integer ℓ.

Because of (c) we call these equivalent conditions **Calabi-Yau condition of the Kähler cone**.

Theorem Let S be a toric Sasaki manifold satisfying Calabi-Yau condition of the Kähler cone. Then, in the above Proposition, we can take

$$\mathbf{o} = -\frac{1}{m+1}\gamma.$$

Using the above Proposition and Theorem we apply the volume minimization argument of Martelli-Sparks-Yau in the following setting.

Let S be a toric Sasaki manifold satisfying Calabi-Yau condition of the Kähler cone. We regard

$$\begin{aligned} \Xi_{\mathbf{o}} &:= \{\xi' \in \mathcal{C}^* \subset \mathfrak{t} \mid \langle \xi', \mathbf{o} \rangle = 1\} \\ &= \{\xi' \in \mathcal{C}^* \subset \mathfrak{t} \mid \langle \xi', \gamma \rangle = -m-1\} \end{aligned}$$

as the space of Reeb vector fields satisfying the Calabi-Yau conditions of the Kähler cone.

Let $\gamma_1, \cdots, \gamma_k$ be basic Kähler classes with respect to the Reeb vector field ξ.

Let $\mathcal{P}_{\xi,1}, \cdots, \mathcal{P}_{\xi,k}$ be compact convex polytopes corresponding to $\gamma_1, \cdots, \gamma_k$, which are assumed to be subsets in the contact moment polytope P_ξ of S,

and $\mathcal{C}_{\xi,1}, \cdots, \mathcal{C}_{\xi,k}$ be convex polyhedral cones in the contact moment convex cone $\mathcal{C}_\xi$ of the Kähler cone $C(S)$ of S such that $\mathcal{P}_{\xi,\alpha} = \mathcal{C}_{\xi,\alpha} \cap P_\xi$.

Choose $\xi' \in \Xi_{\mathbf{o}}$, and set for $\alpha = 1, \cdots, k$

$$\mathcal{P}_{\xi'} = \{p \in \mathcal{C}_\xi \mid \langle \xi', p \rangle = 1\},$$

$$\mathcal{P}_{\xi',\alpha} = \mathcal{C}_{\xi,\alpha} \cap P_{\xi'},$$

$$\Delta_{\xi',\alpha} = \{p \in \mathcal{C}_{\xi,\alpha} \mid \langle \xi', p \rangle \leq 1\}.$$

We now consider the functional $W : \Xi_{\mathbf{o}} \to \mathbb{R}$ defined by

$$\begin{aligned} W(\xi') &:= \sum_{\alpha=1}^{k} \log \frac{\mathrm{Vol}(\mathcal{P}_{\xi',\alpha})}{|\xi'|} \\ &= \sum_{\alpha=1}^{k} \log((m+1)\mathrm{Vol}(\Delta_{\xi'})). \end{aligned}$$

Theorem Let S be a toric Sasaki manifold with Calabi-Yau condition of the Kähler cone, i.e. $c_1^B(S) > 0$ and $c_1(D) = 0$.

(1) W is a strictly convex function on $\Xi_{\mathbf{o}}$.

(2) If we have a critical point $\xi' \in \Xi_{\mathbf{o}}$ such that

$$\mathcal{P}_{\xi'} = \mathcal{P}_{\xi',1} + \cdots + \mathcal{P}_{\xi',k}$$

then there exist transverse couple Kähler-Einstein metrics with respect to ξ'.

(3) In the case of $k = 1$, if we take $\gamma_1 = c_1^B(S)$ and $\mathcal{P}_{\xi,1} = \mathcal{P}_\xi$ then we have $\mathcal{P}_{\xi',1} = \mathcal{P}_{\xi'}$ for any $\xi' \in \Xi_{\mathbf{o}}$, and thus the assumption in (2) is always satisfied. Further, the functional W is a strictly convex proper function and always have a critical point. (This case is due to Martelli-Sparks-Yau.)

Even if we assume

$$2\pi c_1^B(S) = (\gamma_1 + \cdots + \gamma_k)$$

and

$$\mathcal{P}_\xi = \mathcal{P}_{\xi,1} + \cdots + \mathcal{P}_{\xi,k}$$

we do not in general obtain

$$\mathcal{P}'_\xi = \mathcal{P}_{\xi',1} + \cdots + \mathcal{P}_{\xi',k}$$

for other $\xi' \in \Xi_{\mathbf{o}}$, a decomposition of the basic first Chern class $c_1^B(S,\xi')$ with respect to ξ' in the form

$$2\pi c_1^B(S,\xi') = \gamma'_1 + \cdots + \gamma'_k.$$

The failure of getting a Minkowski sum decomposition can be seen from the non-linearity of the CR f-twist of Apostolov-Calderbank.

If $\xi' \in \mathfrak{t}$ is another Reeb vector field then there is a positive Killing potential f of ξ' with respect to ξ satisfying

$$\xi' = f\xi + K_f$$

where f is a positive affine function and $K_f \in C^\infty(D)$. This implies

$$\eta_{\xi'} = \eta_D(\xi')^{-1}\eta_D = \frac{1}{f}\eta_\xi.$$

If $x^1, \cdots, x^n$ and $x'^1, \cdots, x'^n$ are affine coordinates in terms of a basis of $\mathfrak{t}$ on $\mathcal{P}_\xi$ and $\mathcal{P}_{\xi'}$ respectively such that $\mathbf{o}$ is $(0, \cdots, 0)$ in both of the coordinates then

$$x'^i = \frac{x^i}{f}.$$

This transform is called the CR f-twist (Apostolov-Calderbank).

Let $\widetilde{\mathcal{P}}_\xi$ and $\widetilde{\mathcal{P}}_{\xi,\alpha}$ be the f-twist of $\mathcal{P}_\xi$ and $\mathcal{P}_{\xi,\alpha}$. If $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$ for some $\mathbf{x}_\alpha \in \mathcal{P}_{\xi,\alpha}$ the the f-transform of $\mathbf{x}$ is

$$\begin{aligned}
\widetilde{\mathbf{x}} &= \frac{\mathbf{x}}{f(\mathbf{x})} \\
&= \frac{\mathbf{x}_1 + \cdots + \mathbf{x}_k}{f(\mathbf{x})} \\
&\neq \sum_{i=1}^{k} \frac{\mathbf{x}_i}{f(\mathbf{x}_i)} \\
&= \widetilde{\mathbf{x}}_1 + \cdots + \widetilde{\mathbf{x}}_k
\end{aligned}$$

The inequality above explains the failure of getting a Minkowski sum

$$\widetilde{\mathcal{P}}'_\xi = \widetilde{\mathcal{P}}_{\xi',1} + \cdots + \widetilde{\mathcal{P}}_{\xi',k}$$

by the f-transform.