Adiabatic limit of Kählér-Ricci flows

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Invariance of plurigenera problem

The following conjecture is classical.

Conjecture 1.1

Let $f : X \longrightarrow S$ be a smooth Kăhler family. Then the plurigenra $P_m(X_s)(X_s := f^{-1}(s))$ is locally constant on S.

The conjecture holds if $f : X \longrightarrow S$ is a projective The conjecture holds if the relative dimension $\dim X - \dim S$ is less than or equal to two.

But otherwise the conjecture is wide open.

Kǎhler-Ricci flow

(X, ω_0): a compact Kǎhler manifold, The initial value problem:

$$\partial_t \omega(t) = -\mathrm{Ric}(\omega(t))$$

on $X \times [0, T)$ where T is the maximal existence time for "suitable solutions".

$$\omega(0) = \omega_0$$

is called a Ḱahler-Ricci flow. Our main interests are as follows:

(1) Existence of the long time (possibly singular) solution $\omega(t)$

(2) The variation of Kǎhler-Ricci flows on a smooth family of compact Kǎhler manifolds,

(3) The degeneration of Kǎhler-Ricci flows on the singular fiber of a flat family.

Basic definitions

Definition 1

Let $L \longrightarrow X$ be a holomorphic line bundle on a complex manifold X. h is said to be a singular hermitian metric on L, if there exists a funciton $\varphi \in L^1_{loc}(X)$ and a C^∞-hermitian metric h_0 on L such that

$$h = h_0 \cdot e^{-\varphi}$$

holds. We define the curvature current Θ_h by

$$\Theta_h := \Theta_{h_0} + \partial\bar{\partial}\varphi$$

L is said to be pseudoeffective if there exists a singular hermitian metric h on L such that $\sqrt{-1}\Theta_h$ is a closed positive current on X.

Decomposition of Conjecture 1.1

We decompose the Conjecture 1.1 into the following two conjectures.

Conjecture 1.2

Let $f : X \longrightarrow S$ be a smooth Kắhler family. Suppose that there exists a fiber X_0 such that K_{X_0} is pseudoeffective. Then $K_{X/S}$ is pseudoeffective.

If $f : X \longrightarrow S$ is projective, then Conjecture 1.2 is an easy consequence of the L^2-extension theorem.

Relative volume form on the family

Conjecture 1.3

Let $f : X \longrightarrow S$ be a smooth Kähler family such that the relateive canonical bundle $K_{X/S}$ is pseudoeffective. Then there exists a relative (degenerate) volume form $d\mu_{X/S}$ such that

(1) $d\mu_{X/S}^{-1}$ is a singular hermitian metric on $K_{X/S}$ with semipositive curvature current.

(2) For every $s \in S$, $d\mu_s^{-1} = d\mu_{X/S}^{-1}|X_s$ is an AZD of K_{X_s}, i.e.,

$$H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}) \otimes \mathcal{J}(d\mu_s^{-m})) \simeq H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$$

holds for every $m \geqq 0$.

Conjecure 1.2 + Cpnjecture 1.3 $\Rightarrow$ Conjecture 1.1

Suppose that Conjecture 1.2 and Conjecture 1.3 hold, Let $f : X \longrightarrow S$ be a smooth Kăhler family, Suppose that a fiber X_0 has a pseudoeffective canonical bundle. Then by Conjecture 1.2, we see that $K_{X/S}$ is pseudoeffective and by Conjecture 1.3, there exists a relative volume form $d\mu_{X/S}$ as in Conjecture 1.3. Then by the L^2-extension theorem

$$H^0(X, \mathcal{O}(mK_{X/S}) \otimes \mathcal{J}(d\mu_{X/S}^{(m-1)})) \longrightarrow H^0(X_0, \mathcal{O}_{X_0} \otimes \mathcal{J}(d\mu_0^{(m-1)}))$$

is surjective, shirinking S, if necessarily. Then since

$$H^0(X_0, \mathcal{O}_{X_0} \otimes \mathcal{J}(d\mu_0^{(m-1)})) \simeq H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0}))$$

Conjecture 1.1 holds.

Proof of Conjecture 1.3 in the projective case

If $f : X \longrightarrow S$ is projective then it is is not difficult to prove Conjecture 1.3, We introduce the following invariant volume form on a compact complex manifold with pseudoeffective canonical bundle.

Definition 2

Let X be a smooth projective variety with pseudoeffective K_X. For every $\varepsilon > 0$. We set

$$d\mu_\varepsilon := \sup\{dV \mid -\mathrm{Ric}\, dV + \varepsilon\omega \geqq 0, \int_X dV = 1\}$$

where dV moves in the set of degenerate volume forms on X such that dV^{-1} is a singular hermitian metric on K_X. We set

$$d\mu_{ext} := \lim_{\varepsilon \downarrow 0} d\mu_\varepsilon$$

and call it the extremal measure on X.

We see that the extremal measures varies has plurisubharmonic variation on a projectuve family.

Theorem 3

Let $f : X \longrightarrow S$ be a smooth projective family. Suppose that the relative canonical bundle $K_{X/S}$ is pseudoeffective. We define the relative extremal measure $d\mu_{X/S,ext}$ by

$$d\mu_{X/S,ext}|X_s := d\mu_{ext,s}$$

where $d\mu_{ext.s}$ denote the extremal measure on X_s,
Then $d\mu_{X/S,ext}$ satisfies

$$-\mathrm{Ric}(d\mu_{X/S,ext}) \geqq 0$$

on X.

K ́ahler-Einstein metrics

Let (X, g) be a K ́ahler manifold, i.e., g is a hemitian metric and

$$\omega = \frac{i}{2}\sum g_{j\bar{k}} dz_i \wedge d\bar{z}_k$$

is a closed form. ω is the K ́ahler form of $(X.g)$.
We define the Ricci form of (X, g) by

$$\mathrm{Ric}_g = -i\partial\bar{\partial}\log\det(g_{i\bar{j}})$$

It is clear that Ric_g is a closed form and since $(\det(g_{i\bar{j}}))^{-1}$ is a hermitian metric on $K_X = \det T^*X$,

$$-\frac{1}{2\pi}\mathrm{Ric}_g$$

is a 1-st Chern form of K_X. We denote Ric_g as $\mathrm{Ric}(\omega)$.

Definition 4

A Ḱahler manifold (X, g)th is said to be Ḱahler-Einstein if there exists a constant c such that

$$\mathrm{Ric}(\omega) = c \cdot \omega$$

holds.

By definition if a compact complex manifold X admits a Ḱahler-Einstein metrics, K_X is ample or $-K_X$ is ample, or K_X is numerically 0.

Calabi's conjecture

Calabi's conjecture is that conversely assuming a compact complex manifold X have a definite 1-st Chern class, X, Calabi asked whether X admits a Kăhler-Einstein metric.

Theorem 5

(Aubin-Yau) Let X be a compact complex manifold.

(1) *If K_X is ample, then there exists a Ḱahler form ω such that*

$$-Ric(\omega) = \omega$$

holds and is unique.

(2) *Let X be a compact Ḱahler manifold with numerically trivial K_X, then for any Ḱahler form ω_0, there exists a unique Ḱahler form ω such that $Ric(\omega) = 0$ holds in the same class of ω_0.*

Singular Kähler-Einstein metrics

Although the solution of Calabi's conjecture, it is difficult to study general compact Kʹahler manifolds, because general compact Kʹahler manifolds do not admits definite 1-st Chern classes.
To overcome this difficulty we need to consider the singular Kähler metric or even singular semi-Kähler metrics.

Theorem 6

Let X be a compact complex manifold with $\kappa(X) = \dim X$, i.e.,, X is of general type. Then X admits a unique closed positive $(1,1)$-current such that

(1) *There exists a Zariski open subset U of X such that T is a C^∞-Kʹahler form on U,*

(2) $-\mathrm{Ric}(T) = T$ *on* U,

(3) *T is of minimal singular closed positive current in $2\pi c_1(K_X)$.*

Let X be a smooth projective variety of general type. Then by Theorem 6, there exists a K'ahler-Einstein current on X. Then this defines a closed positive $(1,1)$-current on the canonical model X_{can}.

Problem 7

What is the natural metric on a smooth projective manifold in general ?

Hodge bundles

- X : smooth projective manifold with $\kappa(X) \geqq 0$,
- $f : X \longrightarrow Y$: Iitaka fibration of X.
- $K_{X/Y} = K_X \otimes f^* K_Y^{-1}$: the relative canonical bundle of $f : X \longrightarrow Y$.
- $$L_{X/Y} = \frac{1}{m!} f_* \mathcal{O}_X(m! K_{X/Y}) \in \mathrm{Pic}(Y) \otimes \mathbb{Q}$$

  the Hodge $\mathbb{Q}$-line bundle

Hodge metrics

We define the singular hermitian metric on $L_{X/Y}$ by

$$h_{X/Y}^{m!}(\sigma,\sigma) = \left| \int_{X/Y} (\sigma \wedge \bar{\sigma})^{\frac{1}{m!}} \right|$$

We call $h_{X/Y}$ the Hodge metric on $L_{X/Y}$.
By the curvature computation by P.A. Griffith we see that

$$\sqrt{-1}\Theta_{h_{X/Y}} \geqq 0$$

holds in the sense of current.

Canonical measure

Let $f : X \longrightarrow Y$ be the Iitaka fibration.

Definition 8

A closed positive $(1,1)$-current ω_Y satisfying:

(1) ω_Y is a C^∞-K$\acute{}$ahler form on a npnempty Zariski open subset U on Y.

(2) On U, ω_Y satusfues the equation:

$$-\mathrm{Ric}(\omega_Y) + i\Theta_{h_{X/Y}} = \omega_Y.$$

where $\Theta_{h_{X/Y}}$ denotes the curvature current of $h_{X/Y}$.

(3) Let dV_Ybe the volume form associated with ω_Y. Then $dV_Y^{-1} \otimes h_{X/Y}$ is a singular hermitian metric on $K_Y + L_{X/Y}$ with minimal singularities.

is said to be the twisted K$\acute{}$ahler-Einstein form on Y.

Twisted Kähler-Einstein metrics

Theorem 9

(Song-Tian, T-) Let X be a smooth projective variety with $\kappa(X) \geqq 0$ and let $f : X \longrightarrow Y$ There exists a unique twisted K$\acute{}$ahler-Einstein current on Y.

Definition 10

Let $f : X \longrightarrow Y$ be an Iitaka fibration and let ω_Y be the twisted K$\acute{}$ahler-Einstein current on Y. Let dV_Y denote the volume form of ω_Y. We identify $h^{-1}_{X/Y}$ as an relative vlume form on $f : X \longrightarrow Y$ and we set

$$d\mu_X := (dV_Y)^{-1} \otimes h^{-1}_{X/Y}.$$

We call $d\mu_X$ the canonical measure on X. $d\mu_X$ is a degenerate volume form on X.

Kắhler-Ricci flows on a compact Kahler manifold

Let X be a compact Kahler manifold and let ω_0 be an arbitrary C^∞ Kắler form on X.

$$\begin{cases} \partial_t \omega & = & -\mathrm{Ric}(\omega) & \text{on } X \times [0, T) \\ \omega & = & \omega_0 & \text{on } X \times \{0\} \end{cases}$$

where t is the maximal existence time for C^∞-solution.

Singular Kăhler-Ricci flow

Theorem 11

Let X be a compact Kăhler manifold and let ω_0 is a C^∞-Kăhler form on X Then Kăhler Ricci flow

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t))$$

$$\omega(0) = \omega_0$$

has a long time current solution such that

(1) *There exists a sequecne of positive numbers* $0 = t_{-1} < t_0 < t_1 < \cdots$,

(2) *There exists a nonempty Zariski open subeset* U_m *of* X *such that* $\omega(t)$ *is a smooth Kăhler form on* $U_m \times [t_{m-1}, t_m)$,

(3) $\omega(t)$ *is a closed positive current in* $[\omega_0] + 2\pi c_1(K_X)t$ *with minimal singularity.*

Construction of singular Kähler-Ricci flows

(0) Rewrite the Ricci flow into the parabolic degenerate Monge-Ampère equation.

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + i\partial\bar{\partial}u)^n}{\Omega},$$

where $\omega_t = \omega_0 - (\mathrm{Ric}\,\Omega) \cdot t$.

(1) Construct weak solution, the viscosity solution by using the method [E-G-Z].

(2) Using Kodaira's lemma for closed positive $(1,1)$-current (see below), we obtain C^2-estimate of the solution.

(3) The higher order regularity follows from the general theory.

Big class

Definition 12

Let L be a line bundle on a compact complex manifold X. L is said to be big, if

$$\limsup_{m \longrightarrow \infty} \frac{\dim H^0(X, \mathcal{O}_X(mL))}{\log m} = \dim X$$

holds.

Here is the counterpart for a closed positive current on a compact complex manifold.

Definition 13

Let X be a compact Kähler manifold. Let $PE(X) \subset H^{1,1}(X, \mathbb{R})$ be the cone of de Rham cohomology class of closed positive currents. $PE(X)$ is said to be **pseudoeffective cone**. A cohomology class $\theta \in PE(X)$ is said to be big class, if θ is in the interior of $PE(X)$.

Closed positive current with minimal singularities

Proposition 5.1

Let X be a compact complex manifold and let θ be a pseudoeffective class. Then there exists a closed positive current T_0 such that for every closed positive current T in θ, there exists a function $f : X \longrightarrow [0, \infty)$ such taht

$$T_0 - T = i\partial\bar{\partial}f$$

holds.

We call the above T_0 a closed positive current with **minimal singularity** in θ. The construction of such a current is an easy consequence of a classical theorem of P. Lelong which asserts that the uppersemicontinuous envelope of the supremum of a family of PSH functions uniformly bouded from above is again plurisubharmonic.

Kodaira's lemma

The original Kodaira's lemma asserts that a big line bundle admits a singular hermitian metric with strictly positive curvature current.

Lemma 14

Let X be a smooth projective manifold and let L be a big line bundle on X. Then there exists an effective $\mathbb{Q}$-divisor E such that $L - E$ is an ample $\mathbb{Q}$-divisor.
In particular L admits a singular hermitian metric with strictly positive curvature current.

The proof is as follows. Let H be a smooth very ample diisor then

$$H^0(X, \mathcal{O}_X(mL - H)) \to H^0(X, \mathcal{O}_X(mL)) \to H^0(H, \mathcal{O}_H(mL))$$

is exact. Hence if $m >> 1$, $H^0(X, \mathcal{O}_X(mL - H)) \neq 0$. This implies the lemma.

Kodaira's lemma in Kähler case

Theorem 15

(Kodaira's lemma for compact Kăler manifolds) Let X *be a compact Kähler manifold and let* $\theta \in H^{1,1}(X, \mathbb{R})$ *be a big class. Thne there exists a strictly positive closed* $(1,1)$*-current* T *such that*

(1) $[T] = \theta$,

(2) T *has an algebraic singularities, i.e., there exists a modification* $\mu : \tilde{X} \to X$ *such that for every* $x \in X$, *there exist a neighbourhood* U *of* x *such that holomorphic functions* $\{g_\ell\}$ *on* U *a positive number* c *such that*

$$\mu^* T | U = c \cdot \log \sum |g_\ell|^2 + O(1)$$

Theorem 16

(Demailly-Paun 2004) Let X be a compact complex manifold equipped with a hermitian metric ω. Let $T = \alpha + i\partial\bar{\partial}\psi$ be a closed $(1,1)$-current on X, where α is smooth and ψ is a quasi-plurisubharmonic function. Assume that $T \geqq \gamma$ for some real $(1,1)$-form γ on X with real coefficients. Then there exists a sequence $T_k = \alpha + i\partial\bar{\partial}\psi_k$ of closed $(1,1)$-currents such that

(i) *ψ_k (and thus T_k) is smooth on the complement $X \supset Z_k$ of an analytic set Z_k, and the $\{Z_k\}$ form an increasing sequence*

$$Z_0 \subset Z_1 \subset \cdots \subset Z_k \subset X$$

(ii) *There is a uniform estimate $T_k \geqq \gamma - \delta_k\omega$ with $\lim_{\downarrow} \delta_k = 0$ as k tends to $+\infty$.*

(iii) *The sequence $\{\psi_k\}$ is nonincreasing, and we have $\lim_{k\longrightarrow\infty} \psi_k = \psi$. As a consequence, $\{T_k\}$ converges weakly to T as k tends to $+\infty$.*

(iv) Near Z_k, the potential ψ_k has logarithmic poles, namely, for every $x_0 \in Z_k$, there is a neighborhood U of x_0 such that

$$\psi_k(z) = \lambda_k \log \sum_{\ell} |g_{k,\ell}|^2 + O(1)$$

for suitable holomorphic functions $\{g_{k,\ell}\}$ on U and $\lambda_k > 0$. Moreover, there is a (global) proper modification $\mu_k : X_k \longrightarrow X$ of X, obtained as a sequence of blow-ups with smooth centers, such that ψ_k can be written locally on X_k as

$$\psi_k \circ \mu_k(w) = \lambda_k \sum n_\ell \log |\tilde{g}_\ell|^2 + f(w)$$

where $\{g_\ell\}$ are local generators of suitable (global) divisors D_ℓ on X_k such that $\sum D_\ell$ has normal crossings, n_ℓ are positive integers, and the f's are smooth functions on X_k.

Use of Kodaira's lemma

For the C^2-estimate for u, we apply the same strategy as in [T, 1988].

$$\omega(t) = \omega_0 + t(-\mathrm{Ric}\Omega) + i\partial\bar{\partial}u$$

By Kodaira's lemma there exists a quasi-plurisubharmonic function ψ such that

$$\omega_{t,\psi} := \omega_0 + t(-\mathrm{Ric}\Omega) + i\partial\bar{\partial}\psi$$

is a strictly positive $(1,1)$-current with algebraic singularity. We set

$$v := u - \psi$$

and apply the (weghted) C^2-estimate for v.

Let $f : X \longrightarrow S$ be a smooth Kähler family of compact Kähler manifold of relative dimension n such that X admits a Kähler form. Let ω_0 be a C^∞-Kaähler form. Suppose that $K_{X/S}$ is pseudoeffective. We consider the family of Kähler-Ricci flows

$$\frac{\partial}{\partial t}\omega_s(t) = -\mathrm{Ric}(\omega_s(t)) \qquad \text{on } X_s \times [0,\infty)$$

$$\omega_s(0) = \omega_0|X_s$$

Let $f : X \longrightarrow S$ $\{\omega_s(t)\}$be smooth Kӑhler family as above. We set

$$dV_{X/S}(t)|X_s := \frac{1}{n!}\omega_s(t)^n$$

and consider the differential equation:

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(dV_{X/S}(t))$$

with the initial condition

$$\omega(0) = \omega_0$$

This construction has a defect, i.e., the continuity of ω_s with respect to the parameter s is not obvious.

The case that $f : X \longrightarrow S$ is projective

If $f : X \longrightarrow S$ is projective, the above construction works,

Theorem 17

*(**Semipositivity of KRF**) Suppose that $f : X \longrightarrow S$ is projective and ω_0 is a 1-st Chern form of a positive line bundle on X. Then we see that*

$$\omega(t) \geqq 0$$

holds on X for every $t \in [0, +\infty)$.

The proof uses the psh variation property of Bergman kernel due to Berndtsson and the L^2-extension theorem.

Variation of ε-neighbourhoods

Since the problem is local, we may assume that S is the unit open disk with ceneter $O \in \mathbb{C}$. For $s \in S$ and $\varepsilon > 0$, we set

$$B(s, \varepsilon) := \{t \in \mathbb{C} | |t - s| < \varepsilon\}$$

We may assume that f is defined over the whole $B(z, \varepsilon)$ for all $s \in S$. We set

$$X_s(\varepsilon) = f^{-1}(B(s, \varepsilon)).$$

We consider the family

$$f_\varepsilon : X(\varepsilon) \longrightarrow S$$

$X(\varepsilon) = \cup_{s \in S} X_s(\varepsilon)$

Family of Kăhler-Ricci flow on the fat family

Now we consider the family of Kăhler-Ricci flow $\{\omega(t,\varepsilon)\}$ constructed as in Theorem 11, Then we define $dV_s(\varepsilon)$ on X_s. by $\omega(t,\varepsilon)|X_s$.

Theorem 18

*(**Continuity theorem**) If $f : X \longrightarrow S$ be a smooth projective family and ω_0 is a 1-st Chern form of an ample line bundle on X. Then*

$$\lim_{\varepsilon\downarrow 0}\omega(t,\varepsilon)|X_s = \omega_s$$

holds.

The advantages of the fat family

(1) If we use the fat family, we can forget the deformation of complex structures. In fact we only need to consider the composition of the functions

$$\log|s-t|$$

and the convex function $(c>0)$

$$\chi_c(w)=\begin{cases}0 & (w\leqq\log\varepsilon)\\ c(w-\log\varepsilon) & (w\geqq\log\varepsilon)\end{cases}$$

and $c\uparrow+\infty$.

(2) There are no singular fibers for a fat family.

The proof of Semipositivity of KRF

(1) Prove the semipositivity of KRF for ε-fat family.

(2) Let $\varepsilon \downarrow 0$.

(3) Use The continuity theorem (Theorem 18). This part essentially depends on the L^2-extension theorem.

Theorem 19

Suppose that there is a fiber X_0 such that K_{X_0} is abundant. Then the family of KRF converges to twisted Kăher-Einstein current for all $X_s (s \in S)$. In particular plurigenra is invariant on S.

Variation of KRF for Kähler case

Let $f : X \longrightarrow S$ be a flat Kähler fibration (we assume the total space X is Kähler for simplicity). We assume X is smooth but we do not assume f is a smooth fibration. We also assume that $K_{X/S}$ is pseudoeffective.
Then we may also consider the fat family as above and the compute the variation.
The argument is parallel except the Theorem 18.

Conjecture 6.1

The continuity theorem (Theorem 18) also holds for Kähler case.

Extension across singular fiber

Theorem 20

For a flat projective family $f : X \longrightarrow S$ such that $K_{X/S}$ is pseudoeffective. Suppose that X, S are smooth. Let ω_0 is a smooth Kähler form on X. Then fiberwise Kähler-Ricci flow on the smooth fibers can be extended across the singular fibers and has PSH variation property.