Coupled Ding stability and related topics

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Introduction: coupled Kähler–Einstein metrics and Kähler–Einstein metrics

Coupled Kähler–Einstein metrics

Let X be a Fano manifold, i.e. a smooth projective variety over $\mathbb{C}$ such that $-K_X$ is ample. We assume that the automorphism group of X is discrete.

Suppose that we fix ample $\mathbb{Q}$-line bundles $L_1, \dots, L_k$ such that $-K_X = L_1 + \cdots + L_k$.

A k-tuple of Kähler metrics $(\omega_1, \dots, \omega_k) \in c_1(L_1) \times \cdots \times c_1(L_k)$ is called a **coupled Kähler–Einstein** metric if it satisfies

$$\mathrm{Ric}(\omega_1) = \cdots = \mathrm{Ric}(\omega_k) = \sum_{i=1}^{k} \omega_i .$$

The definition is due to Hultgren–Witt Nyström.
It generalises the Kähler–Einstein metric $\mathrm{Ric}(\omega_1) = \omega_1$ (case when $k = 1$), which we briefly review.

Ding functional for Kähler–Einstein metrics

We fix a reference metric $\omega_0 \in c_1(-K_X)$, and write

$$\mathcal{H} := \{\phi \in C^\infty(X, \mathbb{R}) \mid \omega_\phi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$
$$\cong \{\text{positively curved hermitian metrics } e^{-\phi}h_0 \text{ on } -K_X\}.$$

Recall also that any hermitian metric $e^{-\phi}h_0$ on $-K_X$ defines a volume form $d\mu_\phi$ on X.

ω_ϕ is Kähler–Einstein if and only if it is a critical point of the **Ding functional**

$$\mathscr{D}(\phi) := \mathscr{L}(\phi) - \mathscr{E}(\phi),$$

where

$$\mathscr{L}(\phi) := -\log \int_X d\mu_\phi, \quad \mathscr{E}(\phi) := \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \phi \omega_0^{n-j} \wedge \omega_\phi^j$$

with $V := \int_X c_1(-K_X)^n$.

Test configurations

Definition

A (very ample) **test configuration** $(\mathcal{X}, \mathcal{L})$ of exponent $m \in \mathbb{N}$ for a Fano manifold $(X, -K_X)$ consists of

- a normal variety $\mathcal{X}$ with a flat projective morphism $\pi : \mathcal{X} \to \mathbb{C}$, which is $\mathbb{C}^*$-equivariant,
- a relatively very ample Cartier divisor $\mathcal{L}$ to which the action $\mathbb{C}^* \curvearrowright \mathcal{X}$ linearises,

such that $\pi^{-1}(1) \cong (X, -mK_X)$. The preimage of $0 \in \mathbb{C}$, written $\mathcal{X}_0 := \pi^{-1}(0)$, is called the central fibre.

Note: we can compactify a test configuration to get a family $(\bar{\mathcal{X}}, \bar{\mathcal{L}}) \to \mathbb{P}^1$.

Ding invariant

Definition

Given a test configuration $(\mathcal{X}, \mathcal{L})$, its **Ding invariant** is

$$\mathrm{Ding}(\mathcal{X}, \mathcal{L}) := -\frac{\bar{\mathcal{L}}^{n+1}}{(n+1)m^{n+1}V} - 1 + \mathrm{lct}(\mathcal{X}, D; \mathcal{X}_0),$$

where D is the (unique) $\mathbb{Q}$-divisor with $\mathrm{supp} D \subset \mathcal{X}_0$ and $-m(K_{\bar{\mathcal{X}}/\mathbb{P}^1} + D) \sim_{\mathbb{Q}} \bar{\mathcal{L}}$.

Recall $\mathrm{lct}(\mathcal{X}, D; \mathcal{X}_0) := \sup_{c \in \mathbb{R}} \{(\mathcal{X}, D + c\mathcal{X}_0)$ is sublc$\}$.
A Fano manifold $(X, -K_X)$ is said to be **Ding stable** if $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) \geq 0$ for any test configuration of any exponent, with equality if and only if $(\mathcal{X}, \mathcal{L})$ is trivial.

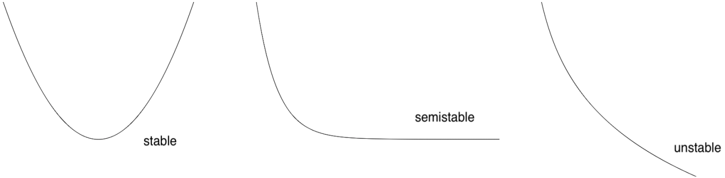
Kähler–Einstein metrics and Ding stability

The work of Berman–Boucksom–Jonsson and Boucksom–Hisamoto–Jonsson (and many other preceding works) show that

$$\mathrm{Ding}(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{\mathscr{D}(\phi_t)}{t}$$

for some "algebraic" geodesic ray $\{\phi_t\}_{t \geq 0} \subset \mathcal{H}$.

Berman–Boucksom–Jonsson used this fact to prove that the (uniform) Ding stability implies the existence of Kähler–Einstein metrics. The geodesic convexity of $\mathscr{D}$ is crucial.



Setup for the coupled Kähler–Einstein case

It is natural to expect that all the results so far extend to the coupled Kähler–Einstien metrics.

- Given $-K_X = L_1 + \cdots + L_k$, pick positively curved hermitian metrics h_i on L_i, with the associated Kähler metric $\theta_i \in c_1(L_i)$ $(i = 1, \ldots, k)$.
- These hermitian metrics define $h'_0 := h_1 \otimes \cdots \otimes h_k$ on $-K_X$, defining the volume form $d\mu'_0$.

We define

$$\boldsymbol{\mathcal{H}} := \mathcal{H}_1 \times \cdots \times \mathcal{H}_k,$$

where

$$\mathcal{H}_i := \{\phi \in C^\infty(X, \mathbb{R}) \mid \theta_i + \sqrt{-1}\partial\bar{\partial}\phi > 0\}.$$

Coupled Ding functional

The following functional was defined by Hultgren–Witt Nyström.

Definition

The **coupled Ding functional** is a map $\mathscr{D}^{\mathrm{cpd}} : \boldsymbol{\mathcal{H}} \to \mathbb{R}$ defined by

$$\mathscr{D}^{\mathrm{cpd}}(\phi_1, \dots, \phi_k) := \mathscr{L}^{\mathrm{cpd}}(\phi_1, \dots, \phi_k) - \sum_{i=1}^{k} \mathscr{E}(\phi_i),$$

where

$$\mathscr{L}^{\mathrm{cpd}}(\phi_1, \dots, \phi_k) := -\log \int_X d\mu'_{\phi_1,\dots,\phi_k},$$

where $d\mu'_{\phi_1,\dots,\phi_k}$ is the volume form corresponding to the product metric

$$e^{-\sum_{j=1}^{k}\phi_j} h'_0 = e^{-\phi_1} h_1 \otimes \cdots \otimes e^{-\phi_k} h_k$$

on $-K_X$.

Properties of the coupled Ding functional

It is known that

- the critical point of $\mathscr{D}^{\mathrm{cpd}}$ is the coupled Kähler–Einstein metric,
- $\mathscr{D}^{\mathrm{cpd}}$ is geodesically convex,

similarly to the usual Kähler–Einstein case.

Hultgren–Witt Nyström conjectured that the existence of coupled Kähler–Einstein metrics is equivalent to the "coupled" version of Ding stability, just as in the usual Kähler–Einstein case.

It seems, however, that their definition of stability needs to be strengthened. This is the main topic of today's talk.

Coupled Ding Stability

Kodaira embeddings for $-K_X = L_1 + \cdots + L_k$

Suppose that we take $m \in \mathbb{N}$ to be sufficiently large and divisible, so that $mL_1, \dots, mL_k$ are all very ample and that the multiplication map

$$H^0(X, mL_1) \otimes \cdots H^0(X, mL_k) \to H^0(X, -mK_X)$$

is surjective.
We thus get the sequence of embeddings

$$\begin{aligned} \iota_{\mathrm{cpd}} : X &\hookrightarrow \mathbb{P}(H^0(X, -mK_X)^\vee) \\ &\hookrightarrow \mathbb{P}(H^0(X, mL_1)^\vee \otimes \cdots \otimes H^0(X, mL_k)^\vee), \end{aligned}$$

in addition to k embeddings

$$\iota_i : X \hookrightarrow \mathbb{P}(H^0(X, mL_i)^\vee)$$

for $i = 1, \dots, k$.

Test configurations and Kodaira embeddings

Proposition (Ross–Thomas)

Any test configuration $(\mathcal{X}_i, \mathcal{L}_i)$ for (X, L_i) of exponent m $(i = 1, \ldots, k)$ can be realised as (the normalisation of) the Zariski closure of $\iota_i(X) \subset \mathbb{P}(H^0(X, mL_i)^\vee)$ under the one-parameter subgroup τ^{A_i} generated by $A_i \in \mathfrak{gl}(H^0(X, mL_i))$.

Thus, given test configurations $(\mathcal{X}_i, \mathcal{L}_i)$ for (X, L_i) of exponent m, we have the generators $A_i \in \mathfrak{gl}(H^0(X, mL_i))$ of $(\mathcal{X}_i, \mathcal{L}_i)$ for all $i = 1, \ldots, k$.
These set of generators define a one-parameter subgroup

$$\tau^{A_1} \otimes \cdots \otimes \tau^{A_k}$$

on $H^0(X, mL_1)^\vee \otimes \cdots \otimes H^0(X, mL_k)^\vee$ by means of the tensor product action.

Coupled test configuraion

Definition

Let $(\mathcal{X}_i, \mathcal{L}_i)$ be a very ample test configuration for (X, L_i) of exponent m with the generator $A_i \in \mathfrak{gl}(H^0(X, mL_i))$ for $i = 1, \dots, k$.
We say that a very ample test configuration $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ is **generated by the $\mathbb{C}^*$-actions of** $(\mathcal{X}_i, \mathcal{L}_i)_{i=1}^k$, if $\mathcal{Y}$ is (the normalisation of) the Zariski closure of $\iota_{\mathrm{cpd}}(X)$ inside

$$\iota_{\mathrm{cpd}} : X \hookrightarrow \mathbb{P}(H^0(X, mL_1)^\vee \otimes \cdots \otimes H^0(X, mL_k)^\vee)$$

with respect to the one-parameter subgroup $\tau^{A_1} \otimes \cdots \otimes \tau^{A_k}$.
$\mathcal{L}_{\mathcal{Y}}$ is the hyperplane bundle.

We can show that $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ indeed defines a test configuration of exponent m for $(X, -K_X)$.

Coupled Ding invariant

Definition

Given a k-tuple of test configurations $(\mathcal{X}_i, \mathcal{L}_i)_{i=1}^k$ as above, its **coupled Ding invariant** is defined as

$$\mathrm{Ding}((\mathcal{X}_i, \mathcal{L}_i)_{i=1}^k)$$
$$:= -\sum_{i=1}^k \frac{\bar{\mathcal{L}}_i^{\,n+1}}{(n+1)m^{n+1}\int_X c_1(L_i)^n} - 1 + \mathrm{lct}(\mathcal{Y}, D_{\mathcal{Y}}; \mathcal{Y}_0),$$

where

- $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ is generated by the $\mathbb{C}^*$-actions of $(\mathcal{X}_i, \mathcal{L}_i)_{i=1}^k$,
- $D_{\mathcal{Y}}$ is the (unique) $\mathbb{Q}$-divisor with $\mathrm{supp} D_{\mathcal{Y}} \subset \mathcal{Y}_0$ and $-m(K_{\bar{\mathcal{Y}}/\mathbb{P}^1} + D_{\mathcal{Y}}) \sim_{\mathbb{Q}} \bar{\mathcal{L}}_{\mathcal{Y}}$.

We then define the coupled Ding stability exactly as the usual Ding stability, using the coupled Ding invariant.

Comparison to Hultgren–Witt Nyström

The original definition of the coupled Ding stability given by Hultgren–Witt Nyström further assumes that $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are all isomorphic.

All previous research on coupled Ding stability (e.g. Hultgren, Hultgren–Witt Nyström, Takahashi) focused on such cases; a particular case intensively studied is when $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are all generated by a single holomorphic vector field (e.g. Delcroix–Hultgren, Futaki–Zhang, Nakamura).

The definition above extends their definition and defines the coupled Ding invariant for a wider class of test configurations.

Slope formula for the coupled Ding functional

Theorem (H. 2021)

Suppose that $\{(\phi_{1,t}, \dots, \phi_{k,t})\}_{t \geq 0} \subset \boldsymbol{\mathcal{H}}$ is a k-tuple of "algebraic" geodesic rays, generated by $A_i \in \mathfrak{gl}(H^0(X, mL_i))$ for $i = 1, \dots, k$.

Let $(\mathcal{X}_i, \mathcal{L}_i)$ be a test configuration for (X, L_i) of exponent m with the generator A_i for $i = 1, \dots, k$.
Then

$$\mathrm{Ding}((\mathcal{X}_i, \mathcal{L}_i)_{i=1}^k) = \lim_{t \to \infty} \frac{\mathscr{D}^{\mathrm{cpd}}(\phi_{1,t}, \dots, \phi_{k,t})}{t}.$$

Corollary

If there exists a coupled Kähler–Einstein metric, $(X; L_1, \dots, L_k)$ is coupled Ding semistable.

Both these results proved by Hultgren–Witt Nyström when $\mathcal{X}_1, \dots, \mathcal{X}_k$ are all isomorphic.

Strengthened coupled Ding stability seems necessary

A finite dimensional approximation (called the **coupled anticanonical balanced metric**) of the coupled Kähler–Einstein metrics was defined by Takahashi.

Theorem (H. 2021)

$(X; L_1, \ldots, L_k)$ admits a coupled anticanonically balanced metric at level m if and only if

$$\mathrm{Ding}((\mathcal{X}_i, \mathcal{L}_i)_{i=1}^k) + \sum_{i=1}^k \mathrm{Chow}_m(\mathcal{X}_i, \mathcal{L}_i) > 0$$

for any k-tuple of nontrivial test configurations of exponent m.

Chow_m is the **Chow weight** that we did not define, but is an algebraic invariant. The "if" part is very unlikely true if we just consider the case when $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are all isomorphic (in this case "only if" was proved by Takahashi).

Key lemma

The key result for the above is the following computation.

Lemma

Let H_i be a positive hermitian form on $H^0(X, mL_i)$, and $\mathrm{FS}_i(H_i)$ be the associated Fubini–Study metric on (X, mL_i) given by the embedding $\iota_i : X \hookrightarrow \mathbb{P}(H^0(X, mL_i)^\vee)$.
Then

$$\mathscr{L}^{\mathrm{cpd}}(\mathrm{FS}_1(H_1), \dots, \mathrm{FS}_k(H_k)) = \mathscr{L}(\mathrm{FS}^{\mathrm{cpd}}(H_1 \otimes \dots \otimes H_k)).$$

where $\mathrm{FS}^{\mathrm{cpd}}$ is the Fubini–Study metric with respect to the embedding $\iota_{\mathrm{cpd}} : X \hookrightarrow \mathbb{P}(H^0(X, mL_1)^\vee \otimes \dots \otimes H^0(X, mL_k)^\vee)$.

This can be checked by direct computation. This formula reduces the computation of the slope of $\mathscr{L}^{\mathrm{cpd}}$ to the one of $\mathscr{L}$, which is well-known (due to Berman–Boucksom–Jonsson).

Open problems

Coupled Yau–Tian–Donaldson conjecture

We would like to prove that the coupled Kähler–Einstein metric exists if and only if $(X; L_1, \ldots, L_k)$ is (uniformly) coupled Ding stable, as originally conjectured by Hultgren–Witt Nyström.

Knowing that the Kähler–Einstein case was solved, it seems natural to adapt the approach of Berman–Boucksom–Jonsson to the coupled case.

There are some problems that appear naturally when we pursue this approach.

List of open problems

1. We defined $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ by means of explicit generators and projective embeddings. This is unsatisfactory. We would like to define $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ in such a way that it depends only on the non-Archimedean metrics defined by $(\mathcal{X}_1, \mathcal{L}_1), \dots, (\mathcal{X}_k, \mathcal{L}_k)$. Is it possible?
2. We also need to define "uniform" coupled Ding stability, which involves the norm of test configurations. Should we use $\|(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})\|$ for the norm, or should it be $\sum_{i=1}^{k} \|(\mathcal{X}_i, \mathcal{L}_i)\|$?
3. We would also like to define the coupled K-energy (work in progress). Note that we can define the coupled cscK (constant scalar curvature Kähler) metrics, after Datar–Pingali.

Coupled δ-invariant

We can define an algebraic invariant δ^{cpd}, which is a coupled version of the δ-invariant defined by Fujita–Odaka.

K. Zhang proved that the coupled Kähler–Einstein metric exists if and only if $\delta^{\mathrm{cpd}} > 1$. Can we prove the uniform coupled Ding stability (appropriately defined) is equivalent to $\delta^{\mathrm{cpd}} > 1$?

For the usual Kähler–Einstein case, $\delta > 1$ is equivalent to the uniform K-stability by using the minimal model programme in birational geometry, but the same method does not seem to naively extend to the coupled case.

If we can prove that the coupled Kähler–Einstein metric exists if and only if the uniform coupled Ding stability holds, we can give a differential-geometric proof of the equivalence between $\delta^{\mathrm{cpd}} > 1$ and the uniform coupled Ding stability, which is purely algebraic.

Remark: the finite dimensional version is indeed true (Rubinstein–Tian–Zhang, H).

Thank you very much for listening!