A decomposition formula for J-stability and its applications

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November 3, 2021

Contents

K-stability and J-stability

Problem in Kähler geometry

When a polarized smooth variety (X, L) has a constant scalar curvature Kähler (f.s. cscK) metric $\omega \in \mathrm{c}_1(L)$?

On the other hand, K-stability is the notion of algebraic geometry and the conjecture below predicts the following;

Yau-Tian-Donaldson conjecture

(X, L) is K-polystable $\Leftrightarrow$ $\exists$cscK metric in $\mathrm{c}_1(L)$.

Similarly, Lejmi-Székelyhidi introduced the algebro-geometric condition called **J-stability** and conjectured that J-stability implies the existence of a certain special Kähler metric.

History of J-stability

Let X^n be a Kähler manifold and χ, ω be Kähler forms on X. Donaldson [Don 99] introduced the notion of **J-flow** (flow of the space of Kähler potentials) and proposed the question:

Question

When J-flow has a stationary solution? Equivalently, when does there exist a solution to the **J-equation**, which is a smooth function φ on X such that

$$\omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$$

is Kähler and

$$\mathrm{tr}_{\omega_\varphi}\chi$$

is the constant $c_0 = \frac{\int_M \chi\wedge\omega^{n-1}}{\int_M \omega^n}$?

Song-Weinkove's criterion

Solvability of J-equation $\Leftrightarrow \exists$ Kähler form ω_0 of the Kähler class $[\omega]$ s.t. the $(n-1,n-1)$-form

$$\left(n\frac{\int_X \chi\wedge\omega^{n-1}}{\int_X \omega^n}\omega_0-(n-1)\chi\right)\wedge\omega_0^{n-2}>0$$

on any point of X.

It is still hard to check the existence of such metric ω_0.
On the other hand, Xiuxiong Chen [Chen 00] introduced the notion of **J-functional** and its gradient flow coincides with J-flow. The $\mathcal{J}_\chi$-functional defined for (1,1)-form χ is

$$\mathcal{J}_\chi(\varphi)=\frac{1}{n!}\int_X\varphi\sum_{k=0}^{n-1}\chi\wedge\omega_0^k\wedge\omega_\varphi^{n-1-k}-\frac{1}{(n+1)!}\int_X c_0\varphi\sum_{k=0}^{n-1}\omega_0^k\wedge\omega_\varphi^{n-k}.$$

Solvability of J-equation is an algebro-geometric condition

Theorem [X.X. Chen 00]

Let X be a Kähler surface. Then, $\left(2\frac{\int_X \chi\wedge\omega}{\int_X \omega^2}\omega - \chi\right)$ is Kähler iff the $\mathcal{J}_\chi$-functional is coercive ($\Longleftrightarrow$ the solution of J-equation exists).

In algebro-geometric perspective, we interpret the theorem of X.X. Chen as follows:

Interpretation of Chen's Theorem

For any pol. sm. surf. (X, L) with an ample line bundle H, if $2\frac{L\cdot H}{L^2}L - H$ is ample, there exists a Kähler form $\omega \in \mathrm{c}_1(L)$ such that $\mathrm{tr}_\omega\chi$ is a constant for any Kähler form $\chi \in \mathrm{c}_1(H)$.

Lejmi-Székelyhidi [LS15] defined **non-Archimedean (NA) J-energy** of test configurations of pol. var. (X, L) as DF-invariant and **J-stability** (cf. §2). They proposed the conjecture in next page. It is a quite generalization of the result of X.X. Chen. This conjecture was proved recently by the combination of works of Gao Chen, Datar-Pingali and Jian Song.

Theorem (Lejmi-Székelyhidi conjecture)

Let (X, L) be a smooth polarized manifold with an ample line bundle H and $\omega \in \mathrm{c}_1(L), \chi \in \mathrm{c}_1(H)$ be Kähler forms. Then the following are equivalent.

1. *A solution to the J-equation exists.*
2. *The J-functional $\mathcal{J}_\chi$ is coercive.*
3. *(X, L) is uniformly J^H-stable.*
4. *(X, L) is uniformly J^H-slope stable.*
5. *(X, L) is J^H-positive.*

- G. Chen proved (1)⇔uniform J-positivity. Datar-Pingali and J. Song proved (1)⇔(5).
- We will define J-stability and J-positivity in §2.

There had been no purely algebro-geometric proof of (3)⇔(4)⇔(5) even when $\dim X = 2$!

Goal in this talk

In the last theorem, (3), (4) and (5) are purely algebro geometric conditions.

- Is there a purely algebro-geometric proof of (5)⇒(3)? ⇒ **Yes!** if X is a surface.
- Uniform J^H-stability ⇔ J^H-stability? ⇒ **No!**
  cf. Uniform K-stability is equivalent to K-stability in (log) Fano case (cf. [Liu-Xu-Zhuang]).

Technical heart

Decompose NA J-energy ($(n+1)$-dimensional intersection number) into the sum of n-dimensional intersection numbers similarly to Mumford-Takemoto slope theory. cf. §3

Main results

Theorem (1)

Let (X, L) be a polarized normal surface and H be a pseudoeffective $\mathbb{Q}$-divisor on X. If

$$2\frac{H \cdot L}{L^2}L - H$$

is nef, then (X, L) is J^H-semistable.

Theorem (2)

(X, L) as above. If H is ample and

$$2\frac{H \cdot L}{L^2}L - H$$

is nef but not ample, then (X, L) is J^H-stable but not uniformly J^H-stable.

Applications of J-stability to K-stability and the problem of cscK metric

DF invariant and K-energy can be decomposed to the sum of the entropy term (>0) and the energy term. We can estimate the energy term of DF (K-energy) of some varieties by applying results of J-stability. Recall the following fact;

Theorem [Chen-Cheng 19]

Assume $\mathrm{Aut}_0(X,L)$ is trivial. Then its K-energy is coercive iff X has a cscK metric in $\mathrm{c}_1(L)$.

Jian-Shi-Song proved the following by Song-Weinkove's criterion for J-stability and the theorem of Chen-Cheng;

Applications of J-stability to K-stability and the problem of cscK metric

Theorem [Jian-Shi-Song 19]

Let $f : (X, H) \to (B, L)$ be a surjective morphism of polarized smooth varieties. Suppose that one of the following holds

1. f is Iitaka fibration (i.e. general fibre of f has a Ricci-flat metric and $f^*L = K_X$).
2. $H = K_X$ and $\dim B = 1$.

Then $(X, H + rL)$ has cscK metrics for $r \gg 0$.

We obtain purely algebro geometric proofs of K-stability of $(X, H + rL)$ when $\dim X = 2$ by applying Theorem (1).

We will give proofs of Theorems (1), (2) in §4.

Contents

Definition of DF invariant and NA J-functional

A **test configuration** $(\mathcal{X}, \mathcal{L})$ over a polarized normal variety (X, L) is a flat, proper, semiample and $\mathbb{C}^*$-equivariant family over $\mathbb{C}$ with general fiber (X, L).

Donaldson-Futaki invariant

Donaldson-Futaki invariant of $(\mathcal{X}, \mathcal{L})$ is

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{1}{L^n}\left(K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{L}^n - \frac{nK_X \cdot L^{n-1}}{(n+1)L^n}\mathcal{L}^{n+1}\right).$$

J-energy functional

Let H be a line bundle. The non-Archimedean $(\mathcal{J}^H)^{\mathrm{NA}}$-energy of $(\mathcal{X}, \mathcal{L})$ is

$$(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{1}{L^n}\left(H_{\mathbb{P}^1} \cdot \mathcal{L}^n - \frac{nH \cdot L^{n-1}}{(n+1)L^n}\mathcal{L}^{n+1}\right).$$

Definition of K-stability and J-stability

Now, we define (X, L) is

- **J^H-semistable** if $(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq 0$ for any test configuration;
- **J^H-stable** if $(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) > 0$ for any non almost trivial test configuration;
- **uniformly J^H-stable** if $\exists \epsilon > 0$ s.t. $(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \epsilon I^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ for any test configuration.

The definition of K-stability is of similar form.

Here, $I^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq 0$ is a "norm" in a sense that $I^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = 0 \Leftrightarrow (\mathcal{X}, \mathcal{L})$ is almost trivial.

If (X, L) is J^{K_X}-semistable, then it is uniformly K-stable.

Definition of flag ideal

Flag Ideal

A **flag ideal** $\mathfrak{a} = \sum_{i=0}^{r} \mathfrak{a}_i t^i$ is a coherent ideal of $\mathcal{O}_{X \times \mathbb{C}}$ such that $\mathfrak{a}_i \subset \mathcal{O}_X$.

It is equivalent to that $\mathfrak{a}$ is $\mathbb{C}^*$-invariant. We assume that $\mathfrak{a}_0 \neq 0$ in this talk.

Test configuration induced by a flag ideal $\mathfrak{a}$

Let $\rho : \mathrm{Bl}_{\mathfrak{a}}(X \times \mathbb{C}) \to X \times \mathbb{C}$ be the blow up along the flag ideal $\mathfrak{a}$. Denote the (semiample) test configuration $(\mathrm{Bl}_{\mathfrak{a}}(X \times \mathbb{C}), \rho^*(L \otimes \mathcal{O}_{\mathbb{C}}) + \rho^{-1}\mathfrak{a})$ by $(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}})$.

To see J-stability, it suffices to check $(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}}) > 0$ for $\mathfrak{a}$.

Slope

Ross-Thomas [RT07] introduced slope stability (analogue of Mumford-Takemoto slope stability). Let $\mathscr{I}_0 \subset \mathcal{O}_X$ be an ideal.

Definition

If $\mathfrak{a} = \mathscr{I}_0 + t$, $(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}})$ is called **NA J-slope energy**.
(X, L) is **uniformly** J^H**-slope stable** if all NA J-slope energy is uniformly positive.

Panov-Ross

2-points blow up of $\mathbb{P}^2$ is slope K-semistable but not K-semistable.

On the other hand,

Uniform J^H-stability $\Leftrightarrow$ uniform slope J^H-stability when H is ample by LS conjecture.

Slope stability is an n-dim condition

If $\mathfrak{a} = \mathcal{O}_X(-D) + t$, ($D$: divisor on X)

$$
\begin{aligned}
&L^{\cdot n}(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}}) \\
&= \left(\frac{n}{n+1} \frac{H \cdot L^{\cdot n-1}}{L^{\cdot n}} \sum_{j=0}^{n} ((\pi^* L^{\cdot n}) - (\pi^* L - D)^{\cdot j} \cdot (\pi^* L)^{\cdot n-j}) \right. \\
&\left. - \sum_{j=0}^{n-1} \pi^* H \cdot ((\pi^* L^{\cdot n-1}) - (\pi^* L - D)^{\cdot j} \cdot (\pi^* L)^{\cdot n-1-j}) \right).
\end{aligned}
$$

- Slope J^H-stability is a condition of subschemes (corresp. to MT theory).
- NA J^H-slope energy is an n-dim intersection number.

Notion of J-positivity

J. Song introduced the notion of (uniform) **J-positivity**. An n-dimensional polarized variety (X, L) is J^H-positive if for any p-dimensional subvariety V of X $(1 \leq p \leq n-1)$,

$$\left(n\frac{H \cdot L^{n-1}}{L^n}L - pH\right) \cdot L^{p-1} \cdot V > 0.$$

Similarly, (X, L) is **uniformly** J^H**-positive** (resp., J^H**-nef**) if there exists $\epsilon > 0$ (resp., ≥ 0) for any p-dimensional subvariety V of X $(1 \leq p \leq n-1)$,

$$\left(n\frac{H \cdot L^{n-1}}{L^n}L - pH\right) \cdot L^{p-1} \cdot V \geq \epsilon L^p \cdot V.$$

If X is a surface and H is ample, J-positivity is equivalent to the ampleness of $2\frac{L \cdot H}{L^2}L - H$.

LS conjecture

(X, L): pol. var. H: ample div. $\omega \in \mathrm{c}_1(L), \chi \in \mathrm{c}_1(H)$: Kähler. TFAE

3. (X, L) is uniformly J^H-stable.
4. (X, L) is uniformly J^H-slope stable.
5. (X, L) is J^H-positive.

Remark

(3): $(n+1)$-dimensional condition.
(4) and (5): n-dimensional condition.

- (3)⇒(4): trivial.
- (4)⇒(5) (2-dim case): For $\mathfrak{a} = \mathcal{O}_X(-\epsilon C) + t$ and $0 < \epsilon \ll 1$,

$$\epsilon^{-1}(L^2)^2(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}_\mathfrak{a}, \mathcal{L}_\mathfrak{a}) = 6(\epsilon C \cdot L)(L \cdot H) - 3(\epsilon C \cdot H)(L^2) - 2\epsilon^2(L \cdot H)(C^2)$$
$$= 3\epsilon \left(2\frac{L \cdot H}{L^2} L - H\right) \cdot C + O(\epsilon^2) \geq 0.$$

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Decomposition Formula

Theorem (Decomposition Formula of NA J-energy)

Let $(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}})$ be a semiample test configuration induced by a flag ideal $\mathfrak{a}$. Then there exists a "good" alteration $\pi : X' \to X$ and $\mathbb{Q}$-divisors D_k on X' s.t. $\overline{\pi^{-1}\mathfrak{a}} = \sum \mathcal{O}_{X'}(-D_k)t^k$ and we can calculate $(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}})$ by using the **mixed multiplicities** of D_k:

$$
\begin{aligned}
&L^{\cdot n}(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}}) \\
&= \frac{1}{\deg \pi} \left(\frac{n}{n+1} \frac{H \cdot L^{\cdot n-1}}{L^{\cdot n}} \sum_{k=0}^{r-1} \sum_{j=0}^{n} ((\pi^* L^{\cdot n}) - (\pi^* L - D_k)^{\cdot j} \cdot (\pi^* L - D_{k+1})^{\cdot n-j}) \right. \\
&\left. - \sum_{k=0}^{r-1} \sum_{j=0}^{n-1} \pi^* H \cdot ((\pi^* L^{\cdot n-1}) - (\pi^* L - D_k)^{\cdot j} \cdot (\pi^* L - D_{k+1})^{\cdot n-1-j}) \right).
\end{aligned}
$$

Mixed Multiplicity

Let $\mathfrak{a}$ be a flag ideal. The following condition is an algebraic and technical condition.

Convexity Condition (*)

$$\mathfrak{a} = \mathcal{O}_X(-D_0) + \mathcal{O}_X(-D_1)t + \cdots + \mathcal{O}_X(-D_{r-1})t^{r-1} + t^r,$$

where each D_i is a Cartier divisor of X. Furthermore, for each $m \in \mathbb{Z}_{\geq 0}$,

$$\mathfrak{a}^m = \sum_{k=0}^{mr} t^k \mathscr{I}_{m,k},$$

where $\mathscr{I}_{m,k} = \mathcal{O}_X(-D_j)^{m-i} \cdot \mathcal{O}_X(-D_{j+1})^i$ for $j = \lfloor \frac{k}{m} \rfloor$ and $i = k - mj$.

Mixed Multiplicity

Theorem

If a test configuration $(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}})$ is a blow up along a flag ideal $\mathfrak{a}$ that satisfies the condition (*), then

$$
\begin{aligned}
&L^{\cdot n}(\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}}) \\
&= \left(\frac{n}{n+1} \frac{H \cdot L^{\cdot n-1}}{L^{\cdot n}} \sum_{k=0}^{r-1} \sum_{j=0}^{n} ((L^{\cdot n}) - (L - D_k)^{\cdot j} \cdot (L - D_{k+1})^{\cdot n-j}) \right. \\
&\left. - \sum_{k=0}^{r-1} \sum_{j=0}^{n-1} H \cdot ((L^{\cdot n-1}) - (L - D_k)^{\cdot j} \cdot (L - D_{k+1})^{\cdot n-1-j}) \right).
\end{aligned}
$$

Alteration

If (*) is not satisfied, by taking a "good" alteration $\pi : X' \to X$ (i.e. π is generically finite and proper), the following hold:

- The integral closure of the inverse image $\mathfrak{a}' = \overline{(\pi \times \mathrm{id}_{\mathbb{C}})^{-1}(\mathfrak{a})}$ satisfies (*).
-
$$\deg \pi (\mathcal{J}^H)^{\mathrm{NA}}(\mathcal{X}_{\mathfrak{a}}, \mathcal{L}_{\mathfrak{a}}) = (\mathcal{J}^{\pi^* H})^{\mathrm{NA}}(\mathcal{X}'_{\mathfrak{a}'}, \mathcal{L}'_{\mathfrak{a}'}),$$

  where $\mathcal{X}'_{\mathfrak{a}'}$ is the blow up of $X' \times \mathbb{C}$ along $\mathfrak{a}'$.

We apply the last theorem to decompose $(\mathcal{J}^{\pi^* H})^{\mathrm{NA}}(\mathcal{X}'_{\mathfrak{a}'}, \mathcal{L}'_{\mathfrak{a}'})$ and obtain the decomposition formula.

Remark on the proof of ∃ alteration

We use the theory of toroidal embeddings and Newton polyhedron to prove ∃ "good" alteration.

Example: How to take a good alteration

Fact (arXiv:2103.04603 §5)

$\mathfrak{a}$ satisfies (*) if "generators of $\mathfrak{a}^m$ form a 1-dim complex".

e.g. $(x \in X) = (0 \in \mathbb{A}^2)$ and $\mathfrak{a} = (t^2, xt, xy)$. In this case,

$$\mathfrak{a}^2 = (t^4, xt^3, x^2t^2, xyt^2, x^2yt, x^2y^2)$$

and the union of bounded faces of $\mathrm{NP}_{\mathfrak{a}}$ is a triangle whose vertices are $(0,0,2)$, $(1,0,1)$ and $(1,1,0)$. However, taking the branch covering, we assume that $x = u^2$ and $y = v^2$. Furthermore, taking the blow up at (u, v), we assume that $s = \frac{u}{v}$ is a regular function and the integral closure of the inverse image of $\mathfrak{a}$ is

$$(t^2, s^2vt, s^4v^2).$$

The union of bounded faces is a segment $[(0,0,2),(4,2,0)]$ and one dimensional.

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Consequences

Theorem (1)

Let (X, L) be a polarized normal surface and H be a pseudoeffective $\mathbb{Q}$-divisor on X. If

$$2\frac{H \cdot L}{L^2}L - H$$

is nef, then (X, L) is J^H-semistable.

Proof Let C be a pseudoeffective $\mathbb{Q}$-Cartier divisor such that

$$(L - C) \cdot H \geq 0.$$

Then, Hodge index theorem shows that

$$2(C \cdot L)(L \cdot H) - (C \cdot H)(L^2) - (L \cdot H)(C^2) \geq 0.$$

The calculation shows that the mixed multiplicities are the sums of these type intersection numbers and nonnegative. Thus, we have Theorem (1).

Corollary

Corollary to (1) (Equivalence of (3), (4) and (5) in LS conjecture)

Let (X, L) be a polarized normal surface and H be a big $\mathbb{Q}$-divisor on X. Then the following are equivalent.

1. (X, L) is uniformly J^H-stable.
2. (X, L) is uniformly J^H-slope stable.
3. (X, L) is uniformly J^H-positive.

Proof It follows from Theorem (1) and from a similar argument of [LS15].

Consequences

Theorem (2)

(X, L) as (1). If H is ample and $2\frac{H\cdot L}{L^2}L - H$ is nef but not ample, then (X, L) is J^H-stable but not uniformly J^H-stable.

Proof If H is big and nef and $L - C$ is pseudoeffective,

$$2(C \cdot L)(L \cdot H) - (C \cdot H)(L^2) - (L \cdot H)(C^2) \geq 0$$

and

$$6(C \cdot L)(L \cdot H) - 3(C \cdot H)(L^2) - 2(L \cdot H)(C^2) > 0.$$

As Theorem (1), we can decompose the $(\mathcal{J}^H)^{\mathrm{NA}}$-energy into these intersection numbers and hence (X, L) is J^H-stable. On the other hand, uniform J^H-stability $\Leftrightarrow$ J^H-positivity of (X, L) by Corollary to (1).

Corollaries

Corollary to (2)

Let (X, L) be a polarized normal surface and H be an ample $\mathbb{Q}$-divisor on X. Then J^H-positivity, uniform J^H-positivity and uniform J^H-stability are equivalent. Furthermore, (X, L) is J^H-stable iff J^H-nef.

There exist a smooth surface (X, L) and an ample line bundle H such that

$$2\frac{H \cdot L}{L^2}L - H$$

is nef but not ample. e.g. $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(e))$ for $e > 0$. Thus, J-nefness $\not\Leftrightarrow$ J-positivity.

Corollary

J-stability and the existence of a unique stationary solution to J-flow are not equivalent.

Stability vs Uniform Stability

There is also an application to log K-stability.

Corollary

There exists a plt pair (X, Δ, L) such that (X, Δ, L) is K-stable but not uniformly K-stable.

For reducible schemes, we obtain the following result.

Corollary

There exists a deminormal surface (X, L) such that (X, L) is K-stable but not uniformly K-stable.

K-stability of fibrations

Adiabatic K-stability

Let $f : (X, H) \to (B, L)$ be a fibration of smooth polarized varieties. (X, H) is **adiabatically K-stable** if $(X, \epsilon H + f^*L)$ is K-stable for $0 < \epsilon \ll 1$.

We obtain the following extensions of [Jian-Shi-Song 19] in a purely algebro-geometric way,

Theorem

Suppose that $f : X \to B$ is a fibered surface and one of the following holds

1. $f : X \to B$ *is an elliptic fibration and* K_X *is nef; or*
2. *There exists a line bundle* L_0 *on* B *such that* $H = K_X + f^*L_0$ *and* $K_X^2 > 0$.

Then (X, H) *is adiabatically K-stable.*

We remark that $(X, \epsilon H + f^*L)$ has also a cscK metric for $0 < \epsilon \ll 1$.

Question

If f is an elliptic fibration, is (X, H) adiabatically K-stable when K_X is not nef?

No! There exist a lot of counter examples to the question. Indeed, if a rational elliptic surface (X, H) admitting at least one "bad" fibre, then it is adiabatically K-unstable. More generally,

Theorem

Let $f : (X, H) \to (C, L)$ be a fibration of pol. var. and D: $\mathbb{Q}$-Cartier divisor on C s.t. $K_X = f^*D$. Suppose that C: curve, M: moduli divisor and B: discriminant div. If (X, H) is adiabatically K-semistable, then (C, B, M, L) is log-twisted K-semistable.

On the other hand, rational elliptic surfaces whose fibre are "good" are adiabatically K-stable and have cscK metrics.

Thank You For Listening!