Collapsed limit spaces may be not collapsed in a synthetic sense.

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Joint work with C. Brena (SNS), N. Gigli (SISSA), X. Zhu (Georgea Tech)

Riemannian convergence theory

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Setting

Gromov-Hausdorff

$$(M_i^n, d_{g_i}) \xrightarrow{GH} (X, d)$$

with "curvature bounds"

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Goal

ⓐ Structure of (X, d)

ⓑ Relationship between X and M_i^n

Riemannian convergence theory

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$$(M_i^n, d_{g_i}) \xrightarrow{GH} (X, d)$$

Gromov-Hausdorff

with "curvature bounds"

① $|\mathrm{Sec}^{g_i}_{M_i^n}| \leq {}^{\exists}K < \infty$

or

② $\mathrm{Sec}^{g_i}_{M_i^n} \geq {}^{\exists}K > -\infty$

or

③ $\mathrm{Ric}^{g_i}_{M_i^n} \geq {}^{\exists}K > -\infty$

etc.

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① $|Sec_{M_i^n}^{g_i}| \leq {}^{\exists}K < \infty$

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Approach

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Synthetic treatment of "curvature bounds" on (X, d)

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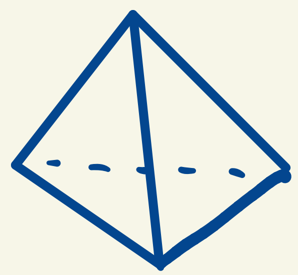
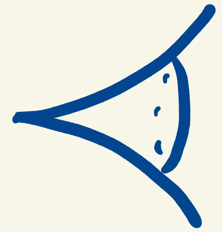
Namely;

ⓐ $^\exists$(Sectional curv $\geq K$ on (X, d))

e.g.  (Alexandrov space of curv. $\geq K$)

ⓑ $^\exists$(Ricci curvature $\geq K$ & dim $\leq N$ on (X, d, m))

e.g. 

Borel measure on X with $\mathrm{supp}\, m = X$

Approach

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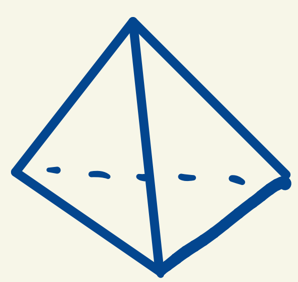
e.g.  (Alexandrov space of curv. $\geq K$)

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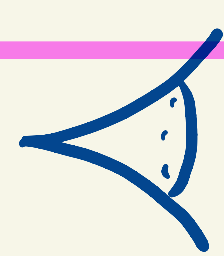
e.g. 

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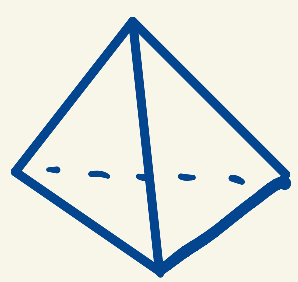
e.g.  (Alexandrov space of curv. $\geq K$)

equivalent in (real) 2-dim. (Lytchak-Stadler)

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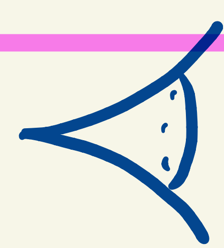
e.g. 

Borel measure on X with $\operatorname{supp} m = X$

Collapsed limit space

Collapsed limit space

Setting

$$\begin{cases} \cdot\ (M_i^n, d_{g_i}) \xrightarrow{GH} (X, d) \\ \cdot\ \mathrm{Ric}^{g_i}_{M_i^n} \geq K g_i \end{cases}$$

$\Rightarrow \dim_H(X, d) \leq n$

Collapsed limit space

Setting

- $(M_i^n, d_{g_i}) \xrightarrow{GH} (X, d)$ ☆
- $Ric^{g_i}_{M_i^n} \geq K g_i$

$\Rightarrow \dim_H(X, d) \leq n$

satisfies "$Ric \geq K$ & $\dim \leq n$"

Def.

① ☆ is non-collapsed if $\dim_H(X, d) = n$

$\overset{\text{Cheeger-Colding}}{\Leftrightarrow}$ $(M_i^n, d_{g_i}, vol_{g_i}) \xrightarrow{mGH} (X, d, \mathcal{H}^n)$ & $\mathcal{H}^n(X) > 0$

measured Gromov-Hausdorff

② ☆ is collapsed if $\dim_H(X, d) < n$

$\overset{\text{Cheeger-Colding}}{\Leftrightarrow}$ $\dim_H(X, d) \leq n-1$

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Conclusion

$$\begin{cases} \cdot\ (M_i^n, d_{g_i}, \mathrm{vol}_{g_i}) \xrightarrow{mGH} (X, d, \mathcal{H}^n) \ \&\ \mathcal{H}^n(X) = 0 \\ \cdot\ (M_i^n, d_{g_i}, c_i \mathrm{vol}_{g_i}) \xrightarrow{mGH} (X, d, {}^{\exists}m) \ \&\ m(X) > 0 \end{cases}$$

$$\left(c_i = \frac{1}{\mathrm{vol}_{g_i} M_i^n} \quad \text{if } X \text{ is compact} \right)$$

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Conclusion

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$$\left(c_i = \frac{1}{\mathrm{vol}_{g_i} M_i^n} \quad \text{if } X \text{ is compact} \right)$$

satisfies "$\mathrm{Ric} \geq K$ & $\dim \leq n$" in a synthetic sense, so-called $\mathrm{RCD}(K, n)$ space

Synthetic treatment of "non-collapsed space with $\mathrm{Ric} \geq k$"

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Setting

$$\begin{cases} \cdot\ (X, d) : \text{complete separable metric sp} \\ \cdot\ \mathfrak{m} : \text{Borel measure on } X \text{ with } \operatorname{supp} \mathfrak{m} = X \\ \cdot\ (X, d, \mathfrak{m}) \text{ satisfies} \\ \qquad \mathrm{Ric} \geq K \ \& \ \dim \leq N \end{cases}$$

for some $K \in \mathbb{R}$ & some $N \in [1, \infty)$ in a synthetic sense

Synthetic treatment of "non-collapsed space with $\mathrm{Ric} \geq K$"

Setting

- (X, d) : complete separable metric sp
- m : Borel measure on X with $\mathrm{supp}\, m = X$
- (X, d, m) satisfies

$$\mathrm{Ric} \geq K \ \& \ \dim \leq N$$

for some $K \in \mathbb{R}$ & some $N \in [1, \infty)$ in a synthetic sense

Def. (De Philippis - Gigli)

(X, d, m) : non-collapsed if $m = c\,\mathcal{H}^N$ $\exists\, c > 0$

$\Rightarrow$ $\exists$ Fine structure on (X, d)

Synthetic treatment of "non-collapsed space with Ric ≥ k"

Setting

- (X, d) : complete separable metric sp
- m : Borel measure on X with $\operatorname{supp} m = X$
- (X, d, m) satisfies

  $\text{Ric} \geq K$ & $\dim \leq N$

  for some $K \in \mathbb{R}$ & some $N \in [1, \infty)$ in a synthetic sense

same!

Def. (De Philippis – Gigli)

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Example from HK metrics on K3 (Sun-Zhang)

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Thm (Sun - Zhang)

(1) $\dim_H (X, d) = 3 \Rightarrow (X, d) \cong T^3/\mathbb{Z}_2$: flat orbifold &

$(K3, d_{g_i}, c_i \, \mathrm{vol}_{g_i}) \xrightarrow{mGH} (X, d, c\mathcal{H}^3)$

: $\mathrm{Ric} \geq 0$ & $\dim \leq 3$

(2) $2 \leq \dim_H (X, d) < 3 \Rightarrow (X, d)$: a singular special Kähler on S^2 &

$(K3, d_{g_i}, c_i \, \mathrm{vol}_{g_i}) \xrightarrow{mGH} (X, d, c \cdot \mathcal{H}^2)$

: $\mathrm{Ric} \geq 0$ & $\dim \leq 2$

Example from HK metrics on K3 (Sun-Zhang)

Setting

- g_i : HK metrics on K3 with diam $= 1$
- $(K3, d_{g_i}) \xrightarrow{GH} (X, d)$

non-collapsed

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Rem.

$$\begin{cases} (1) \quad \dim_H(X,d) = 4 \Rightarrow (X,d) : \text{HK orbifold} \\ \qquad \left(\text{recall}: (K3, d_{g_i}, \mathrm{vol}_{g_i}) \xrightarrow{\text{mGH}} (X, d, H^4)\right) \\ (2) \quad 1 \le \dim_H(X,d) < 2 \Rightarrow (X,d) \cong ([0,1], d_{\mathrm{Eucl}}) \end{cases}$$

$$(K3, d_{g_i}, c_i \mathrm{vol}_{g_i}) \xrightarrow{\text{mGH}} ([0,1], d_{\mathrm{Eucl}}, m)$$

in general $\neq c H^1$

$\Rightarrow$ Odaka-san's talk (c.f. H.-Sun-Zhang)

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$$(K3, d_{g_i}, c_i \mathrm{vol}_{g_i}) \xrightarrow{\text{mGH}} ([0,1], d_{\mathrm{Eucl}}, m)$$

in general $\subset \mathcal{H}^1$, $\not\ll$ (over m)

$\Rightarrow$ Odaka-san's talk (c.f. H.-Sun-Zhang)

$\Rightarrow$ $^\forall$ metric str. of collapsing HK metrics on K3 is an Alex. sp of curv ≥ 0.

Target of the talk and main result

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Is there a nice criterion to check the non-collapsed condition for a given space?

(because non-collapsed spaces have nice structural results)

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Rem

$$\int_X \langle \nabla f_1, \nabla f_2 \rangle \, dm = - \int_X f_1 \Delta f_1 \, dm$$

i.e. for $(M^n, g, e^{-\varphi} dvol_g)$,

$$\Delta f = \operatorname{tr} \mathrm{Hess}_f - \langle \nabla \varphi, \nabla f \rangle$$

Target of the talk and main result

Conj (De Philippis – Gigli)

- (X, d, m) : $Ric \geq K$ & $\dim \leq N$
- $\Delta f = \mathrm{tr}\, \mathrm{Hess}_f \quad \forall f$

$\Rightarrow$ ① $m = c\mathcal{H}^n$, where $n = \dim_{\mathcal{H}}(X, d)$

② $Ric \geq K$ & $\dim \leq n$

Rem

$$\int_X \langle \nabla f_1, \nabla f_2 \rangle \, dm = -\int_X f_1 \Delta f_1 \, dm$$

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$\rightsquigarrow (X, d, \mathcal{H}^n)$: non-collapsed

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($\Leftarrow$ easy)

$\rightsquigarrow (X, d, \mathcal{H}^n)$: non-collapsed

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Rem

$N \in \mathbb{N}$ & $\dim_{\mathcal{H}}(X, d) > N - 1$

$\Rightarrow \begin{cases} \Delta f = \mathrm{tr}\,\mathrm{Hess}_f \quad \forall f \\ N = n \end{cases}$

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($\Leftarrow$ easy)

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$N \in \mathbb{N}$ & $\dim_H(X, d) > N - 1$

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Thm (Brena – Gigli – H. – Zhu)

Conj is true

Key idea

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Find a new proof of

$$\int_X \langle \nabla f_1, \nabla f_2 \rangle \, d\mathcal{H}^n = -\int_X f_1 \operatorname{tr} \operatorname{Hess}_{f_2} \, d\mathcal{H}^n$$

by heat kernel p of $(X, d, \mathfrak{m})$

Key idea

Find a new proof of "unweighted"

$$\int_X \langle \nabla f_1, \nabla f_2 \rangle \, d\mathcal{H}^n = -\int_X f_1 \, \mathrm{tr}\, \mathrm{Hess}_{f_2} \, d\mathcal{H}^n$$

by heat kernel p of (X, d, m)

Rem

We can get even if we start with an weighted

Riem. mfd : $(M^n, g, e^{-f}\mathrm{vol}_g)$, $f \in C^\infty(M^n)$

Proof

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Step 1 Consider

$$\begin{cases} \cdot\ \Phi_t : X \longrightarrow L^2(X, m) \\ \qquad\quad\ \ \rotatebox[origin=c]{-90}{$\in$} \qquad\qquad\quad \rotatebox[origin=c]{-90}{$\in$} \\ \qquad\quad\ \ x \longmapsto (y \mapsto p(x, y, t)), \\ \cdot\ g_t := \Phi_t^* \, g_{L^2}. \end{cases}$$

Proof

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- $\Phi_t : X \longrightarrow L^2(X, m)$
  $x \longmapsto (y \mapsto p(x, y, t))$
- $g_t := \Phi_t^* g_{L^2}$.

Step 2 Prove as $t \to 0^+$;

$$t\, m(B_{\sqrt{t}}(\cdot)) \cdot g_t \longrightarrow c\, \underline{g} \quad \text{in } L^p \quad \forall p < \infty.$$

Riem. metric of (X, d, m) " $d_g = d$ "

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Infinitesimal analogue of Varadhan's asym.:

$$-4t \log p(x, y, t) \to d(x, y)^2 \quad (t \downarrow 0)$$

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Riem. metric of (X, d, m) "$d_g = d$"

Infinitesimal analogue of Varadhan's asym.:

$$-4t \log p(x, y, t) \to d(x, y)^2 \ (t \downarrow 0)$$

cannot be improved to $p = \infty$ because of $\exists$ singular pt in general

Step 3

Prove

$$\nabla^{\gamma} g_t = -\frac{1}{4}\, d \,\Delta\, p(x, x, 2t)$$

Step 3 Prove

$$\nabla^* g_t = -\frac{1}{4} d \Delta p(x, x, 2t)$$

Final step

$$\nabla^* \left(t^{\frac{n+2}{2}} g_t \right) = -\frac{1}{4} d \Delta t^{\frac{n+2}{2}} p(x, x, 2t)$$

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$\downarrow$ as $t \to 0^+$

0 by Gaussian est. for p

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$$\underbrace{\nabla^*\left(t^{\frac{n+2}{2}} g_t\right)}_{\downarrow} = \underbrace{-\frac{1}{4} d \Delta t^{\frac{n+2}{2}} p(x, x, 2t)}_{\downarrow \text{ as } t \to 0^+}$$

$$c \cdot \nabla^*\left(\frac{dH^n}{dm} g\right) \qquad 0 \quad \text{by Gaussian est. for } p$$

(recall: $t\, m(B_{\sqrt{t}}(\cdot))\, g_t \to c \cdot g$)

$$\nabla^{\gamma}\left(\frac{dH^{n}}{dm}\, g\right) = 0$$

$$\nabla^{*}\left(\frac{d\mathcal{H}^{n}}{dm}\,g\right) = 0$$

i.e.

$$\int_X g(\nabla f_1, \nabla f_2)\, d\mathcal{H}^n = -\int_X f_1 \operatorname{tr} \operatorname{Hess}_{f_2}\, d\mathcal{H}^n$$

$$\nabla^{*}\left(\frac{d\mathcal{H}^n}{dm}\, g\right) = 0$$

i.e.

$$\int_X g(\nabla f_1, \nabla f_2)\, d\mathcal{H}^n = -\int_X f_1 \operatorname{tr} \mathrm{Hess}_{f_2}\, d\mathcal{H}^n$$

$$\Rightarrow \quad \frac{d\mathcal{H}^n}{dm} = \text{const.} \qquad //$$

$(\Delta f_2 = \operatorname{tr} \mathrm{Hess}_{f_2})$

Cor.

(X, d, m) : $Ric \geq K$ & $\dim \leq N$

TFAE :

① Essential dim. $= N$

② $\dim_H (X, d) = N$

③ $N \in \mathbb{N}$ & $\dim_H X > N - 1$

④ Topological dim. of $X = N$

⑤ ${}^\exists S \subset X$: closed s.t $m(S) = 0$ & $X \setminus S$: top. N-mfd.

⑥ $m \ll \mathcal{H}^N$

⑦ $m = c\,\mathcal{H}^N$

Thank you

very much!