Recent Development in Value Distribution Theory of the Gauss Map of Complete Minimal Surfaces

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Plan

We will give a brief survey of recent studies on value distribution of the Gauss map of complete minimal surfaces in Euclidean space.

1. Meromorphic Functions on $\mathbf{C}$
   - A geometric interpretation for the little Picard theorem
   - Notion of Totally ramified value and its weight
2. Gauss Map of Complete Minimal Surfaces in $\mathbf{R}^3$
   - A geometric interpretation for the Fujimoto theorem
   - Ramification estimate for the Gauss map of algebraic case
3. Gauss Map of Complete Minimal Surfaces in $\mathbf{R}^4$
   - A geometric interpretation for the Fujimoto theorem
   - Ramification estimate for the Gauss map of algebraic case
4. Outstanding Problems

§1. Meromorphic Functions on $\mathbf{C}$

Theorem (Little Picard Theorem)

$f\colon \mathbf{C} \to \overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$: *a nonconstant meromorphic function*
$D_f := \#(\overline{\mathbf{C}} \backslash f(\mathbf{C}))$: *the number of omitted values of* f
Then

$$D_f \leq 2. \quad \textit{(sharp)}$$

Example ($D_f = 2$)

$$f(z) = e^z, \quad \text{omitted values of } f\text{: } 0, \infty \text{ (2 values)}$$

The least upper bound "2" has a geometric interpretation.

A Geometric Interpretation

Theorem (S. S. Chern (1960), Ahlfors, etc.)

$f\colon \mathbf{C} \to \overline{\Sigma}_\gamma$: *a nonconstant holomorphic map*
Here, $\overline{\Sigma}_\gamma$: *a closed Riemann surface of genus* γ
$D_f := \#(\overline{\Sigma}_\gamma \backslash f(\mathbf{C}))$: *the number of omitted values of* f
Then

$$D_f \leq \chi(\overline{\Sigma}_\gamma) = (\textit{The Euler Characteristic of } \overline{\Sigma}_\gamma) = 2 - 2\gamma,$$

such that

- $\gamma = 0$: $D_f \leq 2$ *(Little Picard Theorem),*
- $\gamma = 1$: $D_f = 0$ *(f is surjective),*
- $\gamma \geq 2$: f *does not exist.*

A Generalization of the Picard Theorem

Theorem (R. Nevanlinna (1929), Robinson (1939))

$f\colon \mathbf{C} \to \overline{\mathbf{C}}$: *a nonconstant meromorphic function*
$q \in \mathbf{Z}_+$, $\alpha_1, \ldots, \alpha_q \in \overline{\mathbf{C}}$ *distinct.*
Suppose that all α_j*-points of* f *have multiplicity at least* ν_j*. Then*

$$\sum_{j=1}^{q}\left(1-\frac{1}{\nu_j}\right) \leq 2.$$

If f *does not take a value* α_j*, then we can take* $\nu_j = \infty$ *and* $1-(1/\nu_j) = 1$*.*

Note:
The inequality corresponds to the defect relation in Nevanlinna theory.

Notion of Totally Ramified Value and Its Weight

Definition

$f\colon \Sigma \to \overline{\mathbf{C}}$: a meromorphic function, Here Σ is a Riemann surface.
$\nu\,(\geq 2) \in \mathbf{Z}_{+} \cup \{\infty\}$
$\alpha \in \overline{\mathbf{C}}$ is a totally ramified value of f of order ν
if $f = \alpha$ has no root of multiplicity less than ν.

We regard omitted values as totally ramified values of order ∞ because $\nu = \infty$ means that $f = \alpha$ has no root of any order.

Definition (Varilon (1929))

Same situation as above.
Then the weight for a totally ramified value of f of order ν is definied by

$$1 - \frac{1}{\nu}.$$

By the total weight ν_f of a number of totally ramified values of f, we mean the sum of their weights.

Ramification Estimate

Theorem (Ramification estimate)

$f\colon \mathbf{C} \to \overline{\mathbf{C}}$: a nonconstant meromorphic function
$D_f := \#(\overline{\mathbf{C}} \backslash f(\mathbf{C}))$: the number of omitted values of f
ν_f: the total weight of a number of totally ramified values of f
Then

$$D_f \leq \nu_f \leq 2. \quad \textit{(sharp)}$$

Example

The Weierstrass $\wp$-function has exactly 4 totally ramified values of order 2

$$e_1 := \wp(\omega/2), \quad e_2 := \wp(\omega'/2), \quad e_3 := \wp((\omega+\omega')/2), \quad \infty.$$

Here the lattice $L = \mathbf{Z}\omega + \mathbf{Z}\omega'$. Thus $\nu_\wp = 4(1-(1/2)) = 2$.

In this case, the least upper bound for ν_f coincides with D_f.

References

- 宮岡礼子著，極小曲面（共立叢書 現代数学の潮流），共立出版, 2022 年.
- 川上裕，藤森祥一著，極小曲面論入門，SGC ライブラリ 147，サイエンス社，2019 年.
- A. Alarcón, F. Forstnerič, F. J. López, Minimal Surfaces from a Complex Analytic Viewpoint, Springer, 2021

Basic Facts of Minimal Surfaces

Σ: a connected, oriented real 2-dimensional C^∞-manifold,
$X : \Sigma \to \mathbf{R}^n$: a conformal minimal immersion.
Then Σ may be considered as a Riemann surface.

ds^2: the induced metric from the Euclidean metric of $\mathbf{R}^n$

Theorem

$$X(\Sigma) \text{ is minimal} \quad \Longleftrightarrow \quad \Delta_{ds^2} X \equiv 0.$$

Corollary

There exists no compact minimal surface without boundary in $\mathbf{R}^n$.

Enneper-Weierstrass Representation

Σ: an open Riemann surface
$\omega = h dz$: a holomorphic 1-form, $g\colon \Sigma \to \overline{\mathbf{C}}$ a meromorphic function
Assume that

- the poles of g of order k coincides exactly with the zeros of ω of order $2k$ (Regularity condition)
- $\phi_1 := (1-g^2)\omega$, $\phi_2 := i(1+g^2)\omega$, $\phi_3 := 2g\omega$.
  $\forall \gamma \in H_1(\Sigma; \mathbf{Z})$, $\mathrm{Re}\int_\gamma \phi_i = 0$ $(i = 1, 2, 3)$ (Period condition)

Then

$$X = \mathrm{Re}\int (\phi_1, \phi_2, \phi_3) : \Sigma \to \mathbf{R}^3$$

is a conformal minimal immersion whose Gauss map is the map g.

$X(\Sigma)$ is minimal $\iff$ The Gauss map g is meromorphic.

Enneper-Weierstrass Representation, continued

We call the pair $(\omega = h dz, g)$ the Weierstrass data (W-data) of $X(\Sigma)$.

- induced metric from $\mathbf{R}^3$: $ds^2 = (1+|g|^2)^{\boxed{2}}|\omega|^2$,
- Gaussian curvature w.r.t ds^2: $K_{ds^2} := -\dfrac{4|g'|^2}{|h|^2(1+|g|^2)^4} \leq 0$,
- Total curvature w.r.t ds^2 $(z = u + iv)$

$$\tau(X) := \int_\Sigma K_{ds^2} dA = -\int_\Sigma \frac{4|g'|^2}{(1+|g|^2)^2} du \wedge dv = -\int_\Sigma g^* \omega_{\mathsf{F.S.}}.$$

$|\tau(X)|$ is the area of $g(\Sigma)$ w.r.t. the metric induced from the Fubini-Study metric $\omega_{\mathsf{F.S.}}$ of $\overline{\mathbf{C}}$.

> Geometric properties of minimal surfaces can be represented by complex analytic data.

Problem

Problem

Given a complete minimal surface, not a plane,
what can be said about the size of the set of points on $\overline{\mathbf{C}}$
omitted by its Gauss map?

- $(X(\Sigma), ds^2)$ is complete if $\int_\gamma ds$ diverges for every differentiable divergent path γ on Σ.
- $X(\Sigma)$ is a plane $\Longleftrightarrow$ The Gauss map g is constant and $K_{ds^2} \equiv 0$.

The Fujimoto Theorem

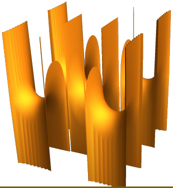
Theorem (Fujimoto (1988, 1992))

$X : \Sigma \to \mathbf{R}^3$ *a complete conformal minimal immersion*
$g : \Sigma \to \overline{\mathbf{C}}$ *its Gauss map*
$D_g := \#(\overline{\mathbf{C}} \backslash g(\Sigma))$*: the number of omitted values of* g
ν_g*: the total weight of a number of totally ramified values of* g
Then

$$D_g \leq \nu_g \leq 4. \quad \textit{(sharp)}$$

Scherk surface

$\Sigma =$ the universal covering of $\overline{\mathbf{C}} \backslash \{\pm 1, \pm i\}$
W-data $(\omega, g) = (4dz/(z^4 - 1), -z)$. Thus $D_g = 4$.



A Geometric Interpretation

Theorem (cf. K. (2013))

Σ: *an open Riemann surface with the conformal metric*

$$ds^2 := (1+|g|^2)^{\boxed{m}}|\omega|^2. \quad (m \in \mathbf{Z}_+)$$

If the metric ds^2 is complete and g is nonconstant, then g can omit at most $\boxed{m}+2$ distinct values.

Remark

Geometric meaning of "2" in "$\boxed{m}$+2" is the Euler characteristic of the Riemann sphere $\overline{\mathbf{C}}$.

The least upper bound for D_g: "4" $= \boxed{2} + \chi(\overline{\mathbf{C}})$

Algebraic Minimal Surfaces

Theorem (Huber (1957) and Osserman (1964))

$X : \Sigma \to \mathbf{R}^3$: a complete minimal immersion with finite total curvature
Then it satisfies

- *Σ is conformally equivalent to $\overline{\Sigma}_\gamma \backslash \{p_1, \ldots, p_k\}$, where $\overline{\Sigma}_\gamma$ is a closed Riemann surface of genus γ and $p_1, \ldots, p_k \in \overline{\Sigma}_\gamma$,*
- *The W-data (ω, g) can be extended meromorphically to $\overline{\Sigma}_\gamma$.*

Definition

When the total curvature of a complete minimal surface is finite, the surface is called an algebraic minimal surface.

The Osserman Problem

Theorem (Osserman (1964))

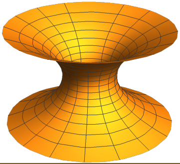
$X\colon \Sigma \to \mathbf{R}^3$: an algebraic minimal surface
$g\colon \Sigma \to \overline{\mathbf{C}}$: its Gauss map
$D_g := \#(\overline{\mathbf{C}}\backslash g(\Sigma))$: the number of omitted values of g
Then

$$D_g \leq 3.$$

Problem (A Survey of Minimal Surfaces, page 90)

Which is the least upper bound for D_g, "2" or "3"?

Example of $D_g = 2$: Catenoid $\Sigma = \mathbf{C}\backslash\{0\}$, $(\omega, g) = (dz/z^2, z)$



Nonexsitence for $D_g = 3$

Theorem (Osserman (1964))

$X\colon \Sigma = \overline{\Sigma}_\gamma \setminus \{p_1, \ldots, p_k\} \to \mathbf{R}^3$: *an algebraic minimal surface*
$g\colon \Sigma \to \overline{\mathbf{C}}$: *its Gauss map with degree* d
If $D_g = 3$, *then*

- $\gamma \geq 1$, *and* $d \geq k \geq 3$,
- *If* $\gamma = 1$, *then* $d = k$ *and each end is an embedded end,*
- $|\tau(X)| \geq 12\pi$.

Theorem

There exist ***NO*** *algebraic minimal surface with* $D_g = 3$ *when*

- $|\tau(X)| = 12\pi$ （*Weitsman and Xavier (1987)*）,
- $|\tau(X)| = 16\pi$ *(Fang (1993)).*

Conjecture A: The least upper bound for D_g is “2”?

Example with $\nu_g = 2.5$ (1)

Conjecture B: The least upper bound for ν_g is "2"?

Conjecture B is not correct.

Example (Miyaoka-Sato (1994) and K. (2006))

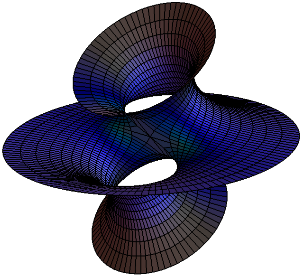
$\Sigma = \mathbf{C}\backslash\{\pm i\}$. The W-data is defined by

$$(\omega, g) = \left(\frac{(z^2+t^2)^2}{(z^2+1)^2}\, dz, \sigma \frac{z^2+1+a(t-1)}{z^2+t} \right), \quad a, t \in \mathbf{R},\ (a-1)(t-1) \neq 0,$$

where $\sigma^2 = (t+3)/a\{(t-1)a+4\}$. For any satisfying $\sigma^2 < 0$, we obtain an algebraic minimal surface of which Gauss map omits 2 values σ, σa. Moreover, $g(0)$ is a totally ramified value of order 2. Since $\deg g = 2$,

$$\nu_g = 1 + 1 + \left(1 - \frac{1}{2}\right) = 2.5.$$

Example with $\nu_g = 2.5$ (2)



Drawn by Prof. Shoichi Fujimori

Another Example with $\nu_g = 2.5$ (1)

Mototsugu Watanabe (my student) finds a new example with $\nu_g = 2.5$!

Example (Watanabe (2022))

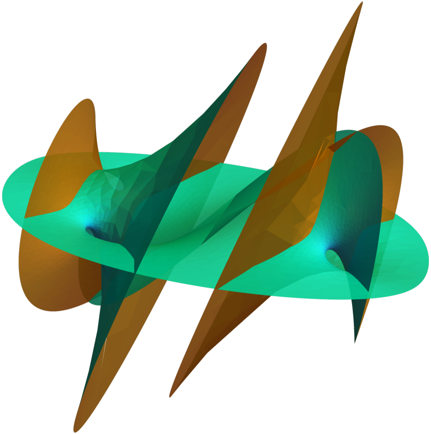
$\Sigma = \mathbf{C}\backslash\{0, \pm i\}$. The W-data is defined by

$$(\omega, g) = \left(\frac{\{(b-a)z^4 + 4(b-1)z^2 + 4(b-1)\}^2}{z^2(z^2+1)^2}\, dz,\ \sigma \frac{(b-a)z^4 + 4a(b-1)z^2 + 4a(b-1)}{(b-a)z^4 + 4(b-1)z^2 + 4(b-1)} \right),$$

where $a, b \in \mathbf{R}\backslash\{1\}$ s.t. $a \neq b$ and $\sigma^2 = (5a + 11b - 16)/(16ab - 11a - 5b) < 0$. Then we obtain algebraic minimal surfaces of which Gauss map omits 2 values σ, σa. Moreover, $\sigma b = g(\pm\sqrt{2}i)$ is a totally ramified value of order 2. Thus

$$\nu_g = 1 + 1 + \left(1 - \frac{1}{2}\right) = 2.5.$$

Another Example with $\nu_g = 2.5$ (2)



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Effective Estimate for D_g and ν_g

Theorem (K-Kobayashi-Miyaoka (2008))

$X\colon \Sigma = \overline{\Sigma}_\gamma \backslash \{p_1, \dots, p_k\} \to \mathbf{R}^3$: *an algebraic minimal surface*
$g\colon \Sigma \to \overline{\mathbf{C}}$: *its Gauss map with degree* d
$D_g := \#(\overline{\mathbf{C}} \backslash g(\Sigma))$: *the number of omitted values of* g
ν_g: *the total weight of a number of totally ramified values of* g
Then

$$D_g \leq \nu_g \leq 2 + \frac{2}{R}, \quad \frac{1}{R} = \frac{\gamma - 1 + (k/2)}{d} < 1.$$

Point: Applying the theory of algebraic curves

- the Riemann-Hurwitz formula (estimate for D_g and ν_g)
- the Riemann-Roch theorem $\to$ the Chern-Osserman inequality (estimate for $1/R$)

Sharpness of KKM-estimate for Some Topological Cases

(1) $(\gamma, k, d) = (0, 3, 2) \cdots \; R^{-1} = (0 - 1 + (3/2))/2 = 1/4$

$$2 + \frac{2}{R} = 2.5$$ (sharp by the Miyaoka-Sato example)

(2) $(\gamma, k, d) = (0, 4, 4) \cdots \; R^{-1} = (0 - 1 + (4/2))/4 = 1/4$

$$2 + \frac{2}{R} = 2.5$$ (sharp by the Watanabe example)

We do not know that this estimate is optimal for all topological cases.

§3. Gauss Map of Complete Minimal Surfaces in $\mathbf{R}^4$

$X = (x^1, x^2, x^3, x^4)\colon \Sigma \to \mathbf{R}^4$: a conformal minimal immersion
Set $\phi_i = \partial x^i$ $(i = 1, 2, 3, 4)$
If we set

$$\omega = \phi_1 - i\phi_2, \quad g_1 = \frac{\phi_3 + i\phi_4}{\phi_1 - i\phi_2}, \quad g_2 = \frac{-\phi_3 + i\phi_4}{\phi_1 - i\phi_2},$$

then ω is a holomorphic 1-form, g_1 and g_2 are meromorphic functions on Σ.
In particular, $G = (g_1, g_2)\colon \Sigma \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ coincides with the Gauss map of $X(\Sigma)$ in $\mathbf{R}^4$.
Furthermore, the induced metric from $\mathbf{R}^4$ is represented as

$$ds^2 = (1 + |g_1|^2)^{\boxed{\ 1\ }}(1 + |g_2|^2)^{\boxed{\ 1\ }}|\omega|^2.$$

A Geometric Interpretation

Theorem (Aiyama-Akutagawa-Imagawa-K (2016))

Σ*: an open Riemann surface with the conformal metric*

$$ds^2 = \prod_{i=1}^{n} (1 + |g_i|^2)^{m_i} |\omega|^2,$$

where $G = (g_1, \ldots, g_n) \colon \Sigma \to (\overline{\mathbf{C}})^n = \underbrace{\overline{\mathbf{C}} \times \cdots \times \overline{\mathbf{C}}}_{n}$ *is a holomorphic map,* ω *is a holomorphic 1-from on* Σ *and each* m_i $(i = 1, \ldots, n)$ *is a positive integer.*

Assume that $g_{i_1}, \ldots, g_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq n)$ *are nonconstant and the others are constant. If the metric is complete and each* g_{i_l} $(l = 1, \cdots, k)$ *omits* $q_{i_l} > 2$ *distinct values, then we have*

$$\sum_{l=1}^{k} \frac{m_{i_l}}{q_{i_l} - 2} \geq 1.$$

Corollary: the Fujimoto theorem

Corollary (Fujimoto (1988))

$X\colon \Sigma \to \mathbf{R}^4$: *a complete nonflat minimal immersion*
$G = (g_1, g_2)\colon \Sigma \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$: *its Gauss map*

(1) *Assume that* g_1 *and* g_2 *are nonconstant and omit* q_1 *and* q_2 *values respectively. If* $q_1 > 2$ *and* $q_2 > 2$, *then we have*

$$\frac{\boxed{\mathit{1}}}{q_1 - 2} + \frac{\boxed{\mathit{1}}}{q_2 - 2} \geq 1.$$

(2) *If either* g_1 *or* g_2, *say* g_2 *is constant, then* g_1 *can omit at most* 3 *values.*

Note: *These results (1) and (2) are optimal.*

Algebraic Minimal Surfaces in $\mathbf{R}^4$

Theorem (Huber (1957) and Osserman (1964))

$X : \Sigma \to \mathbf{R}^4$: a complete minimal immersion with finite total curvature
Then it satisfies

- *Σ is conformally equivalent to $\overline{\Sigma}_\gamma \backslash \{p_1, \ldots, p_k\}$, where $\overline{\Sigma}_\gamma$ is a closed Riemann surface of genus γ and $p_1, \ldots, p_k \in \overline{\Sigma}_\gamma$,*
- *The W-data (ω, g_1, g_2) can be extended meromorphically to $\overline{\Sigma}_\gamma$.*

Definition

When the total curvature of a complete minimal surface is finite, the surface is called an algebraic minimal surface.

Effective Estimate for ν_{g_1} and ν_{g_2}

Theorem (Hoffman-Meeks (1980), K. (2009))

$X\colon \Sigma = \overline{\Sigma}_\gamma \backslash \{p_1, \dots, p_k\} \to \mathbf{R}^4$: *an algebraic minimal surface*
$G = (g_1, g_2)\colon \Sigma \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$: *its Gauss map*
d_i: *the degree of* g_i $(i = 1, 2)$
ν_{g_i}: *the total weight of a number of totally ramified values of* g_i $(i = 1, 2)$

(1) *If* g_1 *and* g_2 *are nonconstant, then* $\nu_{g_1} \leq 2$, *or* $\nu_{g_2} \leq 2$, *or*

$$\frac{1}{\nu_{g_1} - 2} + \frac{1}{\nu_{g_2} - 2} \geq R_1 + R_2 > 1, \quad R_i = \frac{d_i}{2\gamma - 2 + k} \quad (i = 1, 2),$$

(2) *If one of* g_1 *and* g_2 *is constant, say* g_2 *is constant, then*

$$\nu_{g_1} \leq 2 + \frac{1}{R_1} \quad \frac{1}{R_1} = \frac{2\gamma - 2 + k}{d_1} < 1.$$

Sharpness of Ramification Estimate

Example (Watanabe (2022))

$\Sigma = \mathbf{C}\backslash\{\pm i\}$. The W-data is defined by

$$(\omega, g_1, g_2) = \left(\frac{(z^2-1)^2}{(z^2+1)^2}\, dz,\ \frac{z^2+a}{z^2-1},\ \frac{z^2+b}{z^2-1} \right),$$

where $a, b \in \mathbf{R}$ satisfy $(a+1)(b+1) = 8$.
Then we obtain algebraic minimal surfaces with $\nu_{g_1} = 2.5$ and $\nu_{g_2} = 2.5$.
This surface is optimal for (1) in the previous estimate
for $(\gamma, k, d_1, d_2) = (0, 3, 2, 2)$. Indeed,

$$R_1 + R_2 = \frac{2}{0-2+3} + \frac{2}{0-2+3} = 4, \quad \frac{1}{\nu_{g_1}-2} + \frac{1}{\nu_{g_2}-2} = 4.$$

We also do not know that this estimate is optimal for all topological cases.

Corollary: Rigidity Theorem (1)

Corollary (Hoffman-Meeks (1980))

$X : \Sigma \to \mathbf{R}^4$: *an algebraic minimal surface*
$G = (g_1, g_2) \colon \Sigma \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$: *its Gauss map*

(1) *If both g_1 and g_2 omit more than 3 values, then $X(\Sigma)$ must be a plane,*

(2) *If one of g_1 and g_2 is constant, say g_2 is constant and if g_1 omits more than 2 values, then $X(\Sigma)$ must be a plane.*

Example (Sharpness for (2) in Corollary)

$\Sigma = \mathbf{C} \backslash \{0\}$. The W-data is defined by

$$(\omega, g_1, g_2) = \left(\frac{dz}{z^2}, z, c \right), \quad c\text{: constant}$$

then we obtain an algebraic minimal surface of which g_1 omits 2 values $0, \infty$.

Corollary: Rigidity Theorem (2)

Proposition (Watanabe (2022))

$X\colon \Sigma = \overline{\Sigma}_0 \backslash \{p_1, \ldots, p_k\} \to \mathbf{R}^4$: an algebraic minimal surface of genus 0
$G = (g_1, g_2)\colon \Sigma \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$: its Gauss map
If both g_1 and g_2 are nonconstant, one of the following holds:

$$\text{(i)}\ \nu_{g_1} \leq 2, \quad \text{(ii)}\ \nu_{g_2} \leq 2, \quad \text{(iii)}\ \frac{1}{\nu_{g_1} - 2} + \frac{1}{\nu_{g_2} - 2} > 2.$$

Corollary (Watanabe (2022))

$X\colon \Sigma = \overline{\Sigma}_0 \backslash \{p_1, \ldots, p_k\} \to \mathbf{R}^4$: an algebraic minimal surface of genus 0
$G = (g_1, g_2)\colon \Sigma \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$: its Gauss map
If both g_1 and g_2 are nonconstant, one of the following holds:

$$\text{(i)}\ D_{g_1} \leq 2, \quad \text{(ii)}\ D_{g_2} \leq 2.$$

Corollary: Rigidity Theorem (3)

Example (Watanabe (2022))

$\Sigma = \mathbf{C}\backslash\{0\}$. The W-data is defined by

$$(\omega, g_1, g_2) = \left(\frac{dz}{z^2},\, az,\, -\bar{a}z\right),$$

where $a \in \mathbf{C}\backslash\{0\}$. Then we obtain algebraic minimal surfaces whose Gauss maps g_1 and g_2 omit 2 values, 0 and ∞ (i.e., $D_{g_1} = D_{g_2} = 2$).

We do not know whether there exists an example with $D_{g_1} = D_{g_2} = 3$ or not.

Outstanding Problem 1: Flat Point Conjecture

Problem 1

If the Gauss map of a complete minimal surface in $\mathbf{R}^3$ has just 4 omitted values, then the Gaussian curvature is strictly negative on everywhere (i.e. the surface has no flat point)?

In other words, a complete minimal surface in $\mathbf{R}^3$ has at least one flat point, then its Gauss map omits at most 3 values.

Note:
This conjecture is true if a complete minimal surface is pseudo-algebraic (this class contains the Schrek surface) because we have

$$D_g \leq 2 + \frac{2}{R} - \frac{l}{d}, \quad \frac{1}{R} = \frac{\gamma - 1 + (k/2)}{d} \leq 1.$$

Here l is the number of (not necessarily totally) ramified values other than omitted values of g.

Outstanding Problem 2: Nonorientable Case

$\widehat{\Sigma}$: a nonorientable Riemann surface, that is,
a nonorientable surface endowed with an atlas whose transition maps are holomorphic and antiholomorphic.
$\pi\colon \Sigma \to \widehat{\Sigma}$: the conformal oriented two sheeted covering of $\widehat{\Sigma}$

Then a conformal map $\widehat{X}\colon \widehat{\Sigma} \to \mathbf{R}^3$ be a nonorientable minimal immersion if $X = \widehat{X} \circ \pi$ is a conformal minimal immersion.

$I\colon \Sigma \to \Sigma$: the antiholomorphic order two deck transformation associated with π
If g is the Gauss map of the surface $X = \widehat{X} \circ \pi$, then we have

$$g \circ I = -\frac{1}{\bar{g}}.$$

Nonorientable Case, continued

Thus there exists a unique map $\hat{g}\colon \widehat{\Sigma} \to \mathbf{RP}^2 \equiv \overline{\mathbf{C}}/\langle I \rangle$ satisfying

$$\hat{g} \circ \pi = p_0 \circ g,$$

where $p_0\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}/\langle I \rangle$ is the natural projection.

$\rightarrow$ We call $\hat{g}$ the generalized Gauss map of $\widehat{X}(\widehat{\Sigma})$.

Theorem (F. J. López and Martín (2000))

The generalized Gauss map of a complete nonorientable minimal surface in $\mathbf{R}^3$ can omit at most 2 points of $\mathbf{RP}^2$.

Nonorientable Case, continued

Note:

- "2" comes from the Fujimoto theorem.
- López and Martín proved that there exist complete nonorientable minimal surface in $\mathbf{R}^3$ whose generalized Gauss map omits 2 points in $\mathbf{RP}^2$.

Problem 2

Are there any complete nonorientable minimal surface with finite total curvature whose generalized Gauss map omits 1 point in $\mathbf{RP}^2$?

Note: From the Osserman theorem, the case of finite total curvature, we know that the generalized Gauss map can omit at most 1 point in $\mathbf{RP}^2$.

Summary

- New example of algebraic minimal surfaces with $\nu_g = 2.5$
  (By Mr. Mototsugu Watanabe)

- A geometric interpretation for D_g and ν_g (Several cases)

- Outstanding Problems
  (The Osserman problem, Flat point conjecture, Nonorientable case)

- (in progress) A geometric interpretation for the maximum number of omitted hyperplanes of the generalized Gauss map of complete minimal surfaces in $\mathbf{R}^n$ (By Ha-K-Watanabe)