MEASURE THEORETIC POSITION ANALYSIS （測度論的位置解析）

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Abstract. This is a report on my talk at the 28-th Complex Geometry Workshop at Kanazawa. In this talk I have introduced a measure theoretic idea on the analysis of position. Typical examples include (i) position analysis related to fundamental domains and (ii) the random projections which appears in the context of the sequence of projective embeddings of algebraic varieties in projective spaces of higher and higher dimensions. I discussed the motivation behind and application to the construction of full-rank holomorphic maps from $\mathbb{C}^n$ to Calabi-Yau manifolds.

1. Random Projection.

Let X be a smooth projective algebraic variety and $L \to X$ an ample line bundle. Let m be a sufficiently large integer so that the complete linear system $|mL| = \mathbb{P}(H^0(X, \mathcal{O}_X(mL)))$ defines a projective embedding

$$\Phi_{|mL|} : X \to \mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{O}_X(mL))^*) \ ,$$
$$X \ni x \mapsto \mathbb{P}(\{\sigma \in H^0(X, \mathcal{O}_X(mL)) \,|\, \sigma(x) = 0\}) \in \mathbb{P}(H^0(X, \mathcal{O}_X(mL))^*) \ .$$

Let $\mathbb{G}(n, |mL|)$ (resp. equivalently $\mathbb{G}(n+1, H^0(X, \mathcal{O}_X(mL))))$ be the Grassmannian consisting of n-dimensional linear subsystems of $|mL|$ (resp. $(n+1)$-dimensional subspace in $H^0(X, \mathcal{O}_X(mL)))$. An element $\mu \in \mathbb{G}(n, |mL|)$ defines an $N-n-1$-dimensional linear subspace $Z^{N-n-1} \subset |mL|^*$, i.e., the projectivization of the subspace of $H^0(X, \mathcal{O}_X(mL))^*$ consisting of linear forms on $H^0(X, \mathcal{O}_X(mL))$ vanishing on holomorphic sections belonging to the n-dimensional subsystem μ. Therefore to each general $\mu \in \mathbb{G}(n, |mL|)$ is associated the center Z^{N-n-1} of the projection s.t. $Z^{N-n-1} \cap X^n = \emptyset$ in $\mathbb{P}^N$ and the linear projection

$$\mu : X \to \mathbb{P}^n$$

is defined in the projective space $|mL|^*$ (modulo projective transformations of $\mathbb{P}^n$) and this is a finite morphism. Let R_μ denote the ramification divisor in X associated to the projection $\mu : X \to \mathbb{P}^n$. Measure Theory naturally appears in this setting. Indeed, suppose that m is large. Then $\dim \mathbb{G}(n, |mL|)$ is large (this is of order m^n). Then the measure concentration phenomenon takes place for the Haar probability measure on $\mathbb{G}(n, |mL|)$. The position of the ramification divisor R_μ in V w.r.to some other geometric object in X independently defined from m can be analyzed a certain Lipschitz function with hopefully uniform Lipschitz constants in m when m becomes large. In this setting, the Lipschitz function under question looks like a constant function taking the mean over $\mathbb{G}(n, |mL|)$. This is the reason why measure theoretic position analysis in the setting of random projection is interesting in complex geometry. We will give two examples related to the measure theoretic analysis of position, one is the position analysis of the fundamental domain and the other is the position analysis in the setting of random projection.

2. First Example.

I think that the most interesting example at present of the measure theoretic position analysis related to fundamental domains is the relationship between between the fundamental domains of a free Fuchsian group and the family of concentric disks $\mathbb{D}(r)$ $(0 < r < 1)$ in the unit disk $\mathbb{D}$. This situation appears in the study of the partition function

$$Z(r) := \sum_{\alpha \in \pi_1(M)} \frac{\mathrm{Area}_{\mathrm{FS}}(F_\alpha \cap \mathbb{D}(r))}{\mathrm{Area}_{\mathrm{FS}}(\mathbb{D}(r))}$$

where M is an algebraic minimal surface in $\mathbb{R}^3$. A much simpler example is the uniformization of elliptic curves based on the fact that an elliptic curve allows self-coverings, which I discuss now. Let E be an elliptic curve and $L \to E$ an ample line bundle. Let $E \to |mL|^*$ denote the projective embedding and let $\mu : E \to \mathbb{P}^1$ be the linear projection associated to $\mu \in \mathbb{G}(1, |mL|)$. Assume that $\deg(\mu) = m = 2d$. Then we have

$$\mu : E \underset{2:1}{\overset{\wp_E}{\to}} \mathbb{P}^1 \underset{d:1}{\overset{A}{\to}} \mathbb{P}^1 .$$

As an elliptic curve allows self-coverings, we can take the best approximation of μ by the Galois covering

$$\nu : E \underset{d:1}{\overset{\pi}{\to}} E' \underset{2:1}{\overset{\wp_{E'}}{\to}} \mathbb{P}^1 .$$

If d is bounded, then the parameter τ in the Gauss's fundamental domain F corresponding to E' also has bounded height $\Re(\tau)$. As d becomes large, then the height $\Re(\tau)$ of τ proportinally becomes large. On the other hand, we have

$$\mathfrak{M}_\tau(\tau) = \frac{1}{|F|_{\mathrm{hyp}}} \int_F \Re(\tau) \frac{dz d\bar{z}}{|z|^2} = \infty$$

where τ corresponds to E'. We interpret this phenomenon in the following way. When we approximate random projections

$$\mu : E \underset{2:1}{\overset{\wp_E}{\to}} \mathbb{P}^1 \underset{d:1}{\overset{A}{\to}} \mathbb{P}^1$$

by Galois coverings

$$\nu : E \underset{d:1}{\overset{\pi}{\to}} E' \underset{2:1}{\overset{\wp_{E'}}{\to}} \mathbb{P}^1 ,$$

the approximation becomes better and better when $m \to \infty$ and we get a sequence of self-coverings

$$E \underset{1:d}{\overset{d:1}{\rightleftarrows}} E'$$

and as $d \to \infty$ $(m \to \infty)$

$$E \underset{1:d}{\leftarrow} E'$$

converges to

$$E \underset{1:\infty}{\leftarrow} \mathbb{C}^*$$

w.r.to the pointed Gromov-Hausdorff distance. This is the uniformization of an elliptic curve. This argument is based on the fact that an elliptic curve allows self-coverings and the simple measure theoretic position analysis of τ (E') in the Gauss's fundamental domain. In the Second Example, I will try to formulate a family version of this argument. References for §1 is [M] (for measure concentration) and [AS] (a remarkable application of measure concentration phenomenon to combinatorics).

3. Second Example.

Let me start with the motivation behind the second example. This is Demailly's algebraic hyperbolicity (see, for instance, [D]). Let X be a smooth projective variety. We say that X is algebraically hyperbolic if the following condition is fulfilled : There exists an $\varepsilon > 0$ s.t. for all compact connected curves $C \subset X$ the inequality

$$\chi(\overline{C}) \geq \varepsilon \deg_\omega C$$

holds, where $\deg_\omega C$ means the degree of C w.r.to a Kähler form on X and $\overline{C}$ is the normalization of C, $-\chi(C) = 2g(\overline{C}) - 2$. For instance, $\mathbb{P}^n$ is not algebraically hyperbolic, because $\mathbb{P}^n$ contains rational curves with arbitrary high degree. The basic question is whether algebraic hyperbolicity is equivalent to the Kobayashi hyperbolicity for projective varieties.

Demailly's algebraic hyperbolicity naturally motivates algebraic non-hyperbolicity. A smooth projective variety X is algebraically non-hyperbolic if there exists a sequence of curves C in X s.t. $\deg_\omega C$ goes to ∞ while $g(\overline{C})$ remain bounded (note that $g(\overline{C})$ is 0 or 1 for non-hyperbolic curves). Therefore, it is essential to find such sequence of curves in the study of algebraically non-hyperbolic projective varieties. I propose an approach to the question of finding a family of elliptic or rational curves of arbitrary high degree from the view point of measure theoretic position analysis.

Let X be a smooth projective variety of dimension ≥ 2 and $L \to X$ an ample line bundle. For sufficiently large m we set $N+1 = \dim H^0(X, \mathcal{O}(mL))$ and we identify $|mL|$ with $\mathbb{P}^N$. We write

$$X \to |mL|^*$$

for the projective embedding $X \ni x \mapsto \{[\sigma] \in |mL|\,; \sigma(x) = 0\} \subset |mL|$. There is a one-to-one correspondence between $\mu \in \mathbb{G}(n, |mL|)$ (i.e., $\mu^n \in |mL|$) and $Z^{N-n-1} \subset |mL|^*$. We identify $\mu \in \mathbb{G}(n, |mL|)$ with the associated linear projection

$$\mu : X \to \mathbb{P}^n$$

with center Z^{N-n-1} in $\mathbb{P}^N$. If μ is chosen so that $X \cap Z = \emptyset$, then $\mu : X \to \mathbb{P}^n$ is a finite holomorphic map. Therefore we have the Riemann-Hurwitz Theorem

$$K_X = \mu^* K_{\mathbb{P}^n} + R_\mu \ ,$$

where R_μ denotes the ramification divisor in X. We ask what is the relationship between the n-dimensional Riemann-Hurwitz Theorem and the 1-dimensional Riemann-Hurwitz Theorem. My proposal to this question is the following :

Proposition 1 (measure theoretic position analysis). *Let (X, L) be a pair of a smooth projective variety X and an ample line bundle L over it.*

(1) Let ε be any small positive number. Let C be a smooth curve in X whose degree w.r.to the polarization mL is of order $O(m^n)$. Then there exists a pair of a $\mu \in \mathbb{G}(n, |mL|)$ and a linear anti-canonical divisor A on $\mathbb{P}^n$ s.t. the following properties are fulfilled :

(i) Any intersection point p of C and R_μ takes place at the regular part $(R_\mu)_{\mathrm{reg}}$.

(ii) The deviation of the angle $\measuredangle([\nu_p C], [\nu_p R_\mu])$ (ν_p being the operation of taking the orthogonal complement and $[\nu_p R_\mu] = \mathrm{Ker}(d\mu)_p$) from the right angle at every intersection point is smaller than ε :

$$\left| \measuredangle([\nu_p C], [\nu_p R_\mu]) - \frac{\pi}{2} \right| < \varepsilon \ .$$

(2) The same situation as in (1). For every intersection point of C and R_μ we can arrange μ in $\mathbb{G}(n, |mL|)$ so that C intersects with R_μ at a point of $(R_\mu)_{\mathrm{reg}}$ and the angle $\measuredangle([\nu_p C], [\nu_p R_\mu])$ is in fact the right angle (i.e., $[\nu_p R_\mu] = \mathrm{Ker}(d\mu)_p$) :

$$\measuredangle([\nu_p C], [\nu_p R_\mu]) = \frac{\pi}{2} \ , \text{i.e., } [\nu_p R_\mu] = \mathrm{Ker}(d\mu)_p \quad \text{for } \forall p \in R_\mu \cap C \ ,$$

where $d\mu$ means the Jacobian of the projection $\mu : X \to \mathbb{P}^n$.

The strange impression of Proposition 1 (2) comes from the nature of its proof. Thew proof is based on a sort of surgery which needs to be localized and therefore it must be performed in a small spacial margin.

Proposition 1 is the basic form of the measure theoretic position analysis. The following is a variant of Proposition 1 :

Let $C \subset X$ be a smooth curve s.t. $\mu(C) = l$ is a line in $\mathbb{P}^n$. We can find a $\mu \in \mathbb{G}(n, |mL|)$ ($m \gg 1$) with the following properties :

We pick $(n-1)$ intersection points $\{p_1, \dots, p_{n-1}\}$ of $B_\mu = \mu(R_\mu)$ and l. Then :

• At any point p in the pre-image of $\{p_1, \dots, p_{n-1}\}$ in C, we have

$$\left| \measuredangle([\nu_p C], [\nu_p R_\mu]) - \frac{\pi}{2} \right| \geq \delta > 0$$

where $\delta > 0$ is a fixed small number.

• For any other intersection point (those which do not belong to the pre-image of $\{p_1, \dots, p_{n-1}\}$) of C and R_μ, the conclusion of Proposition 1 (1) (ii), i.e.,

$$\left| \measuredangle([\nu_p C], [\nu_p R_\mu]) - \frac{\pi}{2} \right| < \varepsilon$$

holds.

We note that the argument in Proposition 1 (2) does not work for intersection points arising from $\{p_1, \dots, p_{n-1}\}$ because the necessary margin needed for the proof of Proposition 1 (2) is too large and therefore the effect of the surgery does not localize.

We consider a linear anti-canonical divisor A consisting of coordinate hyperplanes in $\mathbb{P}^n$ and the special 1-form on $\mathbb{P}^n$ which is holomorphic on $(\mathbb{C}^*)^n = \mathbb{P}^n \setminus \{\text{coordinate hyperplanes}\}$ defined by

$$\zeta := \sum_{j=2}^{n+1} a_j d \log \zeta_{i,j} \quad \left(\zeta_{i,j} := \frac{z_j}{z_i} \right) .$$

The special feature of ζ is the following. If $\{a_j\}_{j=2}^{n+1}$ is generic, then the restriction of ζ to l has logarithmic poles at intersections with coordinate hyperplanes $\{z_i = 0\}_{i=1}^{n+1}$ and therefore has just $(n-1)$ zeros on l (we may assume that these are simple zeros). Moreover, we may suppose that l is a line with the property that the intersection of l and A satisfies the condition $l \cap \{\text{vertices of } A\} = \emptyset$. For a given C as above (i.e., $C \subset X$ is smooth of degree at most of order m^n) and a $\mu \in \mathbb{G}(n, |mL|)$ satisfying the above property, we can find a ζ so that

• The restriction ζ_l of ζ to the line l satisfies that the $(n-1)$ zeros of ζ_l coincides with $\{p_1, \dots, p_{n-1}\}$, i.e.,

$$(\zeta_l)_0 = \{p_1, \dots, p_{n-1}\}$$

holds. Therefore l is tangent to B_μ at $\{p_1, \dots, p_{n-1}\}$.

Proposition 2 (measure theoretic position analysis on a Calabi-Yau manifold). *Let X be a projective algebraic Calabi-Yau manifold and C a smooth curve in X of degree at most of order m^n. Suppose that there exists a $\mu \in \mathbb{G}(n, |mL|)$ s.t. $l = \mu(C)$ is a line in $\mathbb{P}^n$ and satisfies the condition that l is tangent to B_μ at $(n-1)$ points of $(B_\mu)_{\text{reg}}$ and all other intersection points of l and B_μ take place at $(B_\mu)_{\text{reg}}$ and transversal. then $C \subset$ is an elliptic curve.*

Proof. The above discussion implies that there certainly exists a pair of a smooth curve $C \subset X$ of degree $\leq O(m^n)$ and a $\mu \in \mathbb{G}(n, |mL|)$ which satisfies the assumptions of Proposition 2. As ζ_l in the above discussion is determined by n unknowns $\{a_2, \dots, a_{n+1}\}$, there exists such $\{a_2, \dots, a_{n+1}\}$ satisfying the property $(\zeta_l)_0 = \{p_1, \dots, p_{n-1}\}$. This implies that $\mu^* \zeta_l|_C$ satisfies

$$(\mu^* \zeta_l|_C)_0 = C \cap R_\mu$$

and

$$(\mu^* \zeta_l|_C)_\infty = C \cap \mu^* K_{\mathbb{P}^n}^{-1} .$$

Therefore we have

$$\deg(\mu^* \zeta_l|_C) = \sharp\{\text{zeros of } \mu^* \zeta_l|_C\} - \sharp\{\text{poles of } \mu^* \zeta_l|_C\} = \deg(\text{the restriction of } R_\mu - K_{\mathbb{P}^n}^{-1}) .$$

The Riemann-Hurwitz Theorem for $\mu : X \to \mathbb{P}^n$ implies

$$K_X = R_\mu - K_{\mathbb{P}^n}^{-1}$$

and we have assumed that X is Calabi-Yau, i.e., $K_X = \mathcal{O}_X$. Therefore the degree of the meromorphic 1-form $\mu^*\zeta_l|_C$ satisfies $\deg(\mu^*\zeta_l|_C) = 0$. Therefore C is an elliptic curve. □

Let us recall the argument in §2. Let $E \to \mathbb{P}^N$ be a projective embedding of an elliptic curve into $\mathbb{P}^N$ by a complete linear system $|mL|$ and $\mu : E \to \mathbb{P}^1$ a projection onto a line in $\mathbb{P}^N$. Assume that $m = \deg \mu = 2d$. Then there exists an elliptic curve E' (corresponding to a point $\tau \in F$ in the Gauss' fundamental domain) s.t. the Galois covering $E \xrightarrow{\pi} E' \xrightarrow{\wp_{E'}} \mathbb{P}^1$ best approximates the projection $\mu : E \to \mathbb{P}^1$ and the approximation indefinitely improves as m becomes large. The argument in §2 is the procedure of constructing a holomorphic map $\mathbb{C}^* \to E$ by using the fact that a torus is the only closed surface admitting unramified self-coverings (via certain limiting argument in terms of the pointed Gromov-Hausdorff convergence).

We ask what happens if we apply the argument in §2 to families $\{C_t\}$ os elliptic curves in X. Here the parameter t represents that of the family and the argument in §2 appears in the extension in the t-direction. The answer to this question is : Suppose that we find an elliptic curve in X by the method of Proposition 2.14. Then fixing the line $l \in \mathbb{P}^n$ and we try to find $\mu \in \mathbb{G}(n, |mL|)$ so that the conditions of Proposition 2.14 are fulfilled. These conditions are stable under small displacement of μ in $\mathbb{G}(n, |mL|)$.

Note. *This argument needs justification, because μ^*l decomposes into several number of components. We note that $\mu(\mu^*l) = l$ (set theoretically). Suppose that $\mu : X \to \mathbb{P}^n$ is $2 : 1$ locally along every local irreducible component of R_μ (this is the case for generic $\mu \in \mathbb{G}(n, |mL|)$). Under this assumption, the image of two different irreducible components coincides with l and therefore the number of local irreducible components of both μ^*l and R_μ is two and each irreducible component is orthogonal to one irreducible component of R_μ and tangent to the other. Therefore we interpret the conclusion $C \cap R_\mu = (\mu|_C^*\zeta_l)_0$ of Proposition 2 in this sense, i.e., in the sense of local irreducible components of R_μ as well as μ^*l.*

Therefore we can find an embedded $\mathbb{P}^{n-1}$ in $\mathbb{G}(n, |mL|)$ so that C deforms to elliptic curves at least in an open neighborhood of the original C. Therefore we have a family of elliptic curves $\{C_t\}_{t\in\mathbb{P}^{n-1}}$ in X where every C_t is a smooth elliptic curve in X on some Zariski open subset of $\mathbb{P}^{n-1}$. In the case when $n = \dim X > 2$, we may assume that exists a Zariski open subset V of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ s.t. $C_a \cap C_b = \emptyset$ for $(a, b) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \setminus V$ (because we may assume that this is the case at least in a neighborhood of the original C).

Proposition 3. *The argument in§1 for one generically chosen C from the above $\{C_t\}_{t\in\mathbb{P}^{n-1}}$ uniquely extends holomorphically to the family version of argument in §2 for the family $\{C_t\}$ (possibly containing a singular one) of elliptic curves.*

Proof. The only problem we have to settle for proving the extendability of the argument in §2 from one C_t to the family $\{C_t\}$ is the uniqueness of E' ($\tau \in F$) in the approximation $E \xrightarrow{\pi} E' \xrightarrow{\wp_{E'}} \mathbb{P}^1$ of the conformal structure of E. However, this is the case in the argument in §2 which is the consequence from the fact that the moduli space of the conformal structures of elliptic curves is identified with the Gauss' fundamental domain F which is of complex dimension 1. The choice of E' is discrete in F and therefore must uniquely extends w.r.to the variation of the parameter t. □

From the uniqueness of the $\wp_{E'}$ in $E \xrightarrow{\pi} E' \xrightarrow{\wp_{E'}} \mathbb{P}^1$, the argument in §2 to one elliptic curve C_t automatically and holomorphically extends to the family $\{C_t\}$ of elliptic curves. Before we go further in this direction, we briefly explain the reason why the family version of the argument in §2 is reasonable. Along the sequence $\{mL\}_{m=1,2,\dots}$ we consider the sequence of the projection $\mu : X \xrightarrow{|mL|} |mL|^* \to \mathbb{P}^n$ (from the center determined by $\mu \in \mathbb{G}(n, |mL|)$) and therefore the degree of $C_{i,t}$ becomes indefinitely larger as m becomes larger. To imagine what happens, if m is becomes larger, in the case where X is a projective algebraic Calabi-Yau manifold, it is reasonable from the argument in §2 and its family version that we look at the infinite covering $(\mathbb{C}^*)^n \to A^n_{\mathbb{C}}$ where $A^n_{\mathbb{C}}$ is an Abelian variety (in particular, the infinite overlap of the image of $(\mathbb{C}^*)^n$).

Theorem 4. *Let the situation be as in Proposition 3. In particular, X is an n-dimensional projective algebraic Calabi-Yau manifold. Then there exists a surjective (full-rank) holomorphic map $F : \mathbb{C}^* \times \mathbb{P}^{n-1} \to X$.*

Proof. By Proposition 3 there exists a family $\{C_t\}_{t\in\mathbb{P}^{n-1}}$ of elliptic curves in X. By applying the argument in §2, we get a surjective holomorphic map $\mathbb{C}^* \to C_t \subset X$. We can extend the argument in §2 applied to one generic C to the family $\{C_t\}_{t\in\mathbb{P}^{n-1}}$ including C. The parameter space of this family is $\mathbb{P}^{n-1}$. Thus we have a surjective holomorphic map $F : \mathbb{C}^* \times \mathbb{P}^{n-1} \to X$. □

As soon as we find a $\mathbb{P}^{n-1}$-family of deformations of the original C in X, we can apply the family version of the argument (E) just as in Theorem 4 to to get a full-rank holomorphic map $\mathbb{C}^* \times \mathbb{P}^{n-1} \to X$. Pre-composing with $\mathbb{C} \overset{\exp}{\to} \mathbb{C}^*$, we obtain a full-rank holomorphic map $\mathbb{C} \times \mathbb{P}^{n-1} \to X$.

The surjective holomorphic map in Theorem 4 has the following property described in the following Proposition 5. Let denote the Kähler form associated to the Euclidean metric of $\mathbb{C}\times\mathbb{P}^{n-1}$ by $\beta_{\mathbb{C}\times\mathbb{P}^{n-1}} = dd^c|z|^2 + \omega_{\mathrm{FS},\mathbb{P}^{n-1}}^{n-1}$, where $z \in \mathbb{C}$ and $\omega_{\mathrm{FS},\mathbb{P}^{n-1}}$ is the Fubini-Study Kähler form on $\mathbb{P}^{n-1}$ defined by $\omega_{\mathrm{FS},\mathbb{P}^{n-1}} = dd^c \log \|Z\|^2$ $(Z \in \mathbb{C}^n)$.

Proposition 5. *Let the situation be as in Theorem 4. Let $F : \mathbb{C}^* \times \mathbb{P}^{n-1} \to X$ be the surjective holomorphic map in Theorem 4. Let $f : \mathbb{C} \times \mathbb{P}^{n-1} \to X$ be the pre-composition of $F : \mathbb{C}^* \times \mathbb{P}^{n-1} \to X$ and the universal covering map $\mathbb{C} \times \mathbb{P}^{n-1} \overset{\exp\times \mathrm{id}_{\mathbb{P}^{n-1}}}{\longrightarrow} \mathbb{C}^* \times \mathbb{P}^{n-1}$. Let ω_X be any Kähler form on X. Let $\mathbb{D}(r) \subset \mathbb{C}$ be the disk of radius r in $\mathbb{C}$ centered at the origin, i.e., $\mathbb{D}(r) := \{z \in \mathbb{C} \,|\, |z| < r\}$. Then $\int_{\mathbb{B}(r)} F^*\omega_X \wedge \beta_{\mathbb{C}^n}^{n-1}$ grows like r^2. In particular Nevanlinna's characteristic function*

$$T_{F,\omega_X}(r) := \int_0^r \frac{dt}{t} \int_{\mathbb{D}(r)\times\mathbb{P}^{n-1}} F^*\omega_X \wedge \beta_{\mathbb{C}^n}^{n-1}$$

grows like r^2.

Proof. The proof of Theorem 4 implies that the surjective holomorphic map $F : \mathbb{C}^* \times \mathbb{P}^{n-1} \to X$ is the family version of the covering map $\mathbb{C}^* \to C_t$ (C_t being an elliptic curve). As the composition $f : \mathbb{C} \to C_t$ of $\mathbb{C} \to \mathbb{C}^*$ and $\mathbb{C}^* \to C_t$ has the property that $\int_{\mathbb{D}(r)} f^*\omega_X$ grows like r^2 $(\mathbb{D}(r) = \{z \in \mathbb{C} \,|\, |z| < r\})$. We thus have the desired result. □

Let $E^{n-1} \to \mathbb{P}^{n-1}$ (E being an elliptic curve) be a holomorphic map and we consider the pre-composition with the universal covering map $\mathbb{C}^* \to E$ to get a full-rank holomorphic map $\varphi : \mathbb{C}^{n-1} \to \mathbb{P}^{n-1}$. We tne have $\varphi^*\omega_{\mathrm{FS},\mathbb{P}^{n-1}} \le C\beta_{\mathbb{C}^{n-1}}$ ($\beta_{\mathbb{C}^{n-1}} := dd^c|z|^2$, $z \in \mathbb{C}^{n-1}$) for some constant C. Therefore, if we pre-compose $F : \mathbb{C}^* \times \mathbb{P}^{n-1} \to X$ with the universal covering map $\mathbb{C} \to \mathbb{C}^*$ and $\varphi : \mathbb{C}^{n-1} \to \mathbb{P}^{n-1}$, we get $F_\varphi : \mathbb{C}^n \to X$. For F_φ, we have the same conclusion as in Proposition 5 for Nevanlinna's characteristic function

$$T_{F_\varphi,\omega_X} := \int_0^r \frac{dt}{t^{2n-1}} \int_{\mathbb{B}(r)} F_\varphi^*\omega_X \wedge \beta_{\mathbb{C}^n}^{n-1}$$

where $\mathbb{B}(r) = \{z \in \mathbb{C}^n \,|\, |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < r^2\}$ stands for the ball of radius r in $\mathbb{C}^n$.

The full-rank holomorphic map $F : \mathbb{C} \times \mathbb{P}^{n-1} \to X$ in Proposition 5 (or its variant $F : \mathbb{C}^n \to X$) is interpreted as a higher dimensional analogue of the Brody curve (complex line) ([B], [KS]), because the Nevanlinna characteristic function satisfies the similar property as that of Brody curves.

Proposition 6. *Let X be a projective algebraic Calabi-Yau manifold which is not necessarily simply connected. Suppose that a full-rank holomorphic map $F : \mathbb{C}^n \to X$ is constructed by applying the family version of the argument (E) to the family of elliptic curves (possibly including singular ones) parameterized by an algebraic variety of dimension $n-1$ which is dominated by a surjective holomorphic map from an $(n-1)$ dimensional Abelian variety. Suppose that the Jacobian of the holomorphic map $F : \mathbb{C}^n \to X$ never vanishes. Then X is an n-dimensional Abelian variety.*

Proof. We prove Proposition 6 by showing that the holomorphic map $F : \mathbb{C}^n \to X$ must be the universal covering map. Let ω_X be a Calabi-Yau (Ricci-flat Kähler) metric on X. Put $\eta_{\mathbb{C}^n} := \prod_{i=1}^n dz_i$ (translation invariant holomorphic n-form on $\mathbb{C}^n$). Then, the Lebesgue measure of $\mathbb{C}^n$ is $d\mathcal{L} = (-i)^n \eta_{\mathbb{C}^n} \wedge \overline{\eta}_{\mathbb{C}^n}$.

Let η_X be a non-vanishing holomorphic n-form on X (this is unique modulo non-zero constant multiplication). Then

$$\rho(z) := \frac{f^*\eta_X}{\eta_{\mathbb{C}^n}}$$

is a non-vanishing holomorphic function on $\mathbb{C}^n$ and satisfies the relation

$$|\rho(z)|^2 d\mathcal{L} = (\text{constant}) f^*\omega_X^n\ .$$

As X is compact the function $|\rho(z)|$ is a bounded PSH function on $\mathbb{C}^n$. Therefore $\rho(z)$ is a constant function. Therefore, if the zero locus of the Jacobian of F in Theorem 4 is empty, then $f^*\omega_X^n$ is proportional to $(-i)^n\eta_{\mathbb{C}^n} \wedge \overline{\eta_{\mathbb{C}^n}}$. This means that ω_X represents a Euclidean metric. □

We note that the holomorphic function

$$\rho(z) := \frac{f^*\eta_X}{\eta_{\mathbb{C}^n}}$$

is always defined in the setting of Theorem 4 independent of the assumption on the Jacobian of $F : (\mathbb{C}^*)^n \to X$ and it certainly satisfies the relation

$$|\rho(z)|^2 d\mathcal{L} = (\text{constant}) f^*\omega_X^n$$

and

$$\int_{\mathbb{B}(r)} f^*\omega_X^n$$

grows like r^{2n} uniformly with respect to the center of the r-ball $\mathbb{B}(r)$. The holomorphic function $\rho(z)$ is globally defined on $\mathbb{C}^n$ and is not identically zero with

$$\rho^{-1}(0) \neq \emptyset$$

if the ramification divisor $R_F^{\mathbb{C}^n}$ of $F : \mathbb{C}^n \to X$ is non-empty. Therefore $\rho(z)$ is a transcendental holomorphic function unless $\rho(z) = \text{const.}$. The reason why a transcendental function $\rho(z)$ appears is that, if $\rho^{-1}(0) \neq \emptyset$, then the non-uniformity is forced to occur around points $\cap_{j\neq i} H_i$ $(i = 1, \ldots, n)$ in the family version of the argument (E) in the limit procedure as $m \to \infty$ of defining the holomorphic map $F : \mathbb{C}^n \to X$.

4. Perspective.

We can probably apply the argument in §3 (measure theoretic position analysis on a Calabi-Yau manifold) to the case of a Fano manifold. In this setting, I conjecture that a full-rank holomorphic map $(\mathbb{C}^*)^n \to X \setminus D$ exists, where X is a Fano manifold and D is a ("maximally" degenerate) anti-canonical divisor on X.

References

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