Uniformization of compact Sasakian manifolds using basic Higgs bundles

Natsuo Miyatake
Email address 1: natsuo.m.math@gmail.com
Email address 2: n-miyatake@imi.kyushu-u.ac.jp
The 28th Symposium on Complex Geometry (Kanazawa) 2022
Joint work with Hisashi Kasuya (Osaka University)

November 7th, 2022

Harmonic bundles

Let $(E, D) \to (M, g_M)$ be a flat bundle over a Riemannian manifold and $\rho_D : \pi_1(M) \to \mathrm{GL}(r, \mathbb{C})$ the holonomy representation.

Definition (Harmonic metric on a flat bundle)

A Hermitian metric h on (E, D) is said to be a *harmonic metric* if the natural ρ_D-equivariant map $\hat{h} : \tilde{M} \to \mathrm{GL}(r, \mathbb{C})/\mathrm{U}(r)$ is a harmonic map.

Definition (Harmonic bundle)

A *harmonic bundle* over (M, g_M) is a pair $(E, D, h) \to (M, g_M)$ consisting of a flat bundle (E, D) and a harmonic metric h on (E, D).

Theorem (Donaldson'87, Corlette '88)

A flat bundle over a compact Riemannian manifold admits a harmonic metric if and only if it is semisimple.

Kobayashi-Hitchin correspondence between Higgs bundles and harmonic bundles (Simpson'88)

Let (X, ω_X) be a Kähler manifold.

Definition (Hitchin '87, Simpson '88, Higgs bundle)

A *Higgs bundle* over X is a pair (E, θ) consisting of a holomorphic vector bundle $E \to X$ and a holomorphic section θ of $\mathrm{End}E \otimes \bigwedge^{1,0}$ satisfying $\theta \wedge \theta = 0$. We call θ the *Higgs field*.

For a Hermitian metric h, we define a connection D as

$$D := \nabla^h + \theta + \theta^{*h},$$

where:

- $\nabla^h = \partial_h + \bar{\partial}$ is the Chern connection of h,
- $\theta^{*h} : E \to E \otimes \bigwedge^{0,1}$ is the adjoint of θ for the metric h.

If the connection D is flat, then $(E, D, h) \to (X, \omega_X)$ is a harmonic bundle.

Definition (Hermitian-Einstein equation)

The following elliptic PDE for a hermitian metric h is called the *Hermitian-Einstein equation*:

$$\Lambda_{\omega_X} F_D^{\perp} = \Lambda_{\omega_X}(F_{\nabla^h} + [\theta \wedge \theta^{*h}])^{\perp} = 0,$$

where we denote by Λ_{ω_X} the adjoint of $\omega_X \wedge$, and by $F_D^{\perp}$ the trace-free part of the curvature. We call the solution of the Hermitian-Einstein equation *Hermitian-Einstein metric*.

Proposition (Simpson'88)

Suppose that a Kähler manifold (X, ω_X) is compact for simplicity. Let h be a Hermitian-Einstein metric of a Higgs bundle (E, θ) over (X, ω_X). If $c_1(E) = c_2(E) = 0$, then the connection $D = \nabla^h + \theta + \theta^{*h}$ is flat.

Let (X, ω_X) be a compact Kähler manifold.

Theorem (Simpson'88)

A Higgs bundle $(E, \theta) \to (X, \omega_X)$ admits a Hermitian-Einstein metric if and only if (E, θ) is polystable.

Corollary (Kobayashi-Hitchin correspondence of Higgs bundles and harmonic bundles, Simpson'88)

For a Higgs bundle $(E, \theta) \to (X, \omega_X)$, there exists a Hermitian metric h such that a connection $D := \nabla^h + \theta + \theta^{*h}$ is flat if and only if (E, θ) is polystable and $c_1(E) = c_2(E) = 0$.

Remark

Let $(E, D, h) \to (X, \omega_X)$ be a harmonic bundle and $D = \partial_h + \bar{\partial}_h + \theta + \theta^{*h}$ the natural decomposition of D. Then $\bar{\partial}_h \circ \bar{\partial}_h = 0$ and $(E, \bar{\partial}_h, \theta)$ is a Higgs bundle. This follows from Corlette's theorem ('88).

Uniformization

Let G_0 be a real semisimple Lie group with Lie algebra $\mathfrak{g}_0$. We denote by $\mathfrak{g}$ the complexification of the Lie algebra $\mathfrak{g}_0$. Suppose that $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}$$

and that the decomposition satisfies the following for all $p, r \in \{1, 0, -1\}$:

$$\overline{\mathfrak{g}^{p,-p}} = \mathfrak{g}^{-p,p},$$
$$[\mathfrak{g}^{p,-p}, \mathfrak{g}^{r,-r}] \subseteq \mathfrak{g}^{p+r,-p-r},$$
$$(-1)^p \mathrm{Tr}(\mathrm{ad}(\cdot)\mathrm{ad}(\cdot)) \text{ is positive definite on } \mathfrak{g}^{p,-p}.$$

Suppose also that the closed subgroup $K_0 \subseteq G_0$ whose Lie algebra is $\mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$ is compact. Then the Lie group G_0 with the decomposition of the Lie algebra is called the *Hodge group of Hermitian type* (Simpson'88).

Example

Let G_0 and $\mathfrak{g}_0$ be

$$G_0 = \mathrm{SU}(n,1) \coloneqq \{g \in \mathrm{GL}(n+1,\mathbb{C}) \mid gJ({}^t\bar{g}) = J, \det(g) = 1\},$$
$$\mathfrak{g}_0 = \mathfrak{su}(n,1) \coloneqq \{u \in \mathrm{M}(n+1,\mathbb{C}) \mid uJ + J({}^t\bar{u}) = 0, \mathrm{Tr}(u) = 0\}.$$

where J is defined as $J \coloneqq \mathrm{diag}(1,\dots,1,-1)$. The complexified Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ decomposes as

$$\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{0,0}, \text{ where}$$
$$\mathfrak{g}^{-1,1} = \left\{ \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \mid A = (a_1,\dots,a_n)\ a_j \in \mathbb{C}\ (j=1,\dots,n) \right\},$$
$$\mathfrak{g}^{1,-1} = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \mid A = {}^t(a_1,\dots,a_n)\ a_j \in \mathbb{C}\ (j=1,\dots,n) \right\},$$
$$\mathfrak{g}^{0,0} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -\mathrm{Tr}(A) \end{pmatrix} \mid A \in \mathrm{M}(n,n) \right\}.$$

Let G_0 be a Hodge group of Hermitian type with the maximal compact subgroup $K_0 \subseteq G_0$. Let K be the complexification of K_0.

Definition (uniformizing bundle, Simpson'88)

Let (X, ω_X) be a compact Kähler manifold. A pair (P_K, θ) consisting of a holomorphic K-bundle P_K over X and a $P_K \times_K \mathfrak{g}^{-1,1}$-valued holomorphic 1-form θ satisfying $[\theta \wedge \theta] = 0$ is said to be a *uniformizing bundle* if θ is an isomorphism between $T^{1,0}X$ and $P_K \times_K \mathfrak{g}^{-1,1}$.

Definition (uniformizing variations of Hodge structure, Simpson'88)

A *uniformizing variations of Hodge structure* is a pair $((P_K, \theta), P_{K_0} \subseteq P_K)$ consisting of a uniformizing bundle (P_K, θ) and a K_0-subbundle $P_{K_0} \subseteq P_K$ of P_K such that the connection $\nabla + \theta + (-\sigma)(\theta)$ is flat, where ∇ is the canonical connection of the reduction $P_{K_0} \subseteq P_K$, and $\sigma : \mathfrak{g} \to \mathfrak{g}$ is the involution defined as $\sigma(u + \sqrt{-1}v) = u - \sqrt{-1}v$ for $u, v \in \mathfrak{g}_0$.

Example (Hitchin'87)

Let X be a compact connected Riemann surface with genus ≥ 2. We choose a square root $K_X^{1/2}$ of the canonical bundle $K_X \to X$. We define a Higgs bundle $(E, \theta) \to X$ over X as

$$E := K_X^{1/2} \oplus K_X^{-1/2},$$
$$\theta := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where 1 is considered to be a holomorphic 1-form taking values in K_X^{-1}. This Higgs bundle is stable, and the corresponding harmonic bundle defines an $\mathrm{SU}(1,1)$-uniformizing variations of Hodge structure.

Theorem (Simpson'88)

Let X be a compact Kähler manifold and $\widetilde{X} \to X$ the universal covering space. Then $\widetilde{X}$ is isomorphic to a bounded symmetric domain $\mathcal{D}$ if and only if there exists a uniformizing variations of Hodge structure $((P_K, \theta), P_{K_0} \subseteq P_K)$ over X with some Hodge group of Hermitian type G_0 such that $G_0/K_0 = \mathcal{D}$.

Proof.

Suppose that there exists a uniformizing variations of Hodge structure $((P_K, \theta), P_{K_0} \subseteq P_K)$ on X. Then from the G_0-flat bundle $\nabla + \theta + (-\sigma)(\theta)$ and the reduction $P_{K_0} \subseteq P_{G_0}$, we have a $\pi_1(X)$-equivariant map: $f : \widetilde{X} \to G_0/K_0$. Since θ is an isomorphism between $T^{1,0}X$ and $P_K \times_K \mathfrak{g}^{-1,1}$, the differential of f

$$df : T^{1,0}\widetilde{X} \to T^{1,0}(G_0/K_0)$$

is isomorphic at each fiber since it is a pullback of θ by the universal covering map. □

Proof.

(continued from the previous page) In particular, f is a local diffeomorphism. We denote by $\omega_{\widetilde{X}}$ the Kähler metric on $\widetilde{X}$ obtained by pulling back the Kähler metric on G_0/K_0 by f. Then $\omega_{\widetilde{X}}$ is complete since X is compact. Since f is a local isometry from a complete Kähler manifold $(\widetilde{X}, \omega_{\widetilde{X}})$, f is a covering map. Furthermore, since G_0/K_0 is simply connected, f is a diffeomorphism.

Conversely, suppose that there exists an isomorphism $f : \widetilde{X} \to \mathcal{D}$. Then since $\pi_1(X)$ acts on X holomorphically, we have a representation $\pi_1(X) \to \mathrm{Aut}(\mathcal{D})$, where we denote by $\mathrm{Aut}(\mathcal{D})$ the set of automorphisms of $\mathcal{D}$. Set $G_0 = \mathrm{Aut}(\mathcal{D})$. Then we have a uniformizing bundle with the Hodge group G_0. □

Example (Simpson'88)

Let (X, ω_X) be an n-dimensional compact Kähler manifold. We define a holomorphic vector bundle E as

$$E := T^{1,0}X \oplus \mathbb{C},$$

where $\mathbb{C}$ is the trivial bundle. We also define a Higgs field θ on E as

$$\theta := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where 1 is considered to be the identity of $T^{1,0}X \otimes \bigwedge^{1,0} \simeq \mathrm{End}(T^{1,0}X)$.

Example

(continued from the previous page) Suppose that (E, θ) is stable and that

$$\int_X \{2c_2(T^{1,0}X) - c_1(T^{1,0}X)^2\} \wedge \omega_X^{n-2} = 0.$$

Then the Hermitian-Einstein metric on (E, θ) gives a $PU(n,1)$-uniformizing variations of Hodge structure and the universal covering $\widetilde{X}$ is isomorphic to the unit ball in $\mathbb{C}^n$.

Basic Higgs bundles on Sasakian manifolds

Let M be a $2n+1$-dimensional real manifold.

Definition (Sasakian structure, Sasakian manifold)

A *Sasakian structure* on M is a pair $(T^{1,0}, (\xi, \eta))$ consisting of:

- An involutive n-dimensional subbundle $T^{1,0} \subseteq TM \otimes_{\mathbb{R}} \mathbb{C}$ of the complexified tangent bundle satisfying $T^{1,0} \cap T^{0,1} = 0$, where $T^{0,1}$ is defined as $T^{0,1} := \overline{T^{1,0}}$.
- A nowhere vanishing real vector field ξ such that $[\xi, T^{1,0}] \subseteq T^{1,0}$ and that $S \oplus \mathbb{R}\xi = TM$, where $S \subseteq TM$ is a real subbundle defined as $S := TM \cap (T^{1,0} \oplus T^{0,1})$.
- A real one form η such that $\eta(\xi) = 1, \ker \eta = S$ and that $d\eta : T^{1,0} \times T^{0,1} \to \mathbb{C}$ is a positive-definite Hermitian form.

A $2n+1$-dimensional real manifold equipped with a Sasakian structure is called a *Sasakian manifold*.

Example

Let $\mathcal{D}$ be an n-dimensional bounded symmetric domain. Then $S^1(\bigwedge^n T^{1,0}\mathcal{D})$ has a Sasakian structure described as follows: Let G_0 be a Hodge group of Hermitian type such that $G_0/K_0 = \mathcal{D}$. Let $\mathfrak{s} \subseteq \mathfrak{g}_0$ be a subspace defined as $\mathfrak{s} := \mathfrak{g}_0 \cap (\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1})$. We denote by $I : \mathfrak{s} \to \mathfrak{s}$ the complex structure on $\mathfrak{s}$ defined as $I(v + \bar{v}) := \sqrt{-1}v - \sqrt{-1}\bar{v}$ for $v \in \mathfrak{g}^{-1,1}$. Then there exists an element v_I of the center $Z(\mathfrak{k}_0)$ such that $I = \mathrm{ad}(v_I)$.

Example

(continued from the previous page) We define a K_0-invariant metric on $\mathfrak{s}$. By considering the action of K_0 on $\bigwedge^n \mathfrak{g}^{-1,1}$, we have a character $\chi : K_0 \to \mathrm{U}(1)$. Denote by K_0' the closed subgroup $\ker \chi$ of G_0, and by $\mathfrak{k}_0'$ its Lie algebra. Let $\mathfrak{m} \subseteq \mathfrak{g}_0$ be the $2n+1$-dimensional subspace $\mathbb{R}v_I \oplus \mathfrak{s}$. Then $\mathfrak{m}$ is naturally isomorphic to $\mathfrak{g}_0/\mathfrak{k}_0'$. We take a linear map $\eta : \mathfrak{m} \to \mathbb{R}$ so that $\eta(v_I) = 1$, $\ker \eta = \mathfrak{s}$. Then $(\mathfrak{g}^{-1,1}, (v_I, \eta))$ defines a Sasakian structure on $G_0/K_0' \simeq S^1(\bigwedge^n T^{1,0}(G_0/K_0)) = S^1(\bigwedge^n T^{1,0}\mathcal{D})$.

Let M be a Sasakian manifold with the Reeb vector field ξ.

Definition (basic form)

A differential form ϕ on M is said to be a *basic form* if $i_\xi \phi = i_\xi d\phi = 0$.

We denote by $\Omega_B^{p,q}(M)$ the space of (p,q)-basic forms.

Definition (basic vector bundle)

A vector bundle $E \to M$ over a Sasakian manifold M equipped with a family $(U_\alpha, s_\alpha : U_\alpha \to E\,|_{U_\alpha})_{\alpha \in A}$ of local trivializations is said to be a *basic vector bundle* if $\bigcup_{\alpha \in A} U_\alpha = M$ and all transition functions are basic.

Definition (basic holomorphic bundle)

A basic vector bundle E is called a *basic holomorphic bundle* if we can take each transition function to be transversely holomorphic.

A basic holomorphic bundle E naturally admits an operator $\bar{\partial}_E : \Omega_B^{p,q}(E) \to \Omega_B^{p,q+1}(E)$.

Definition (basic Higgs bundle, Biswas-Kasuya'19)

A *basic Higgs bundle* over M is a pair $(E, \theta) \to M$ consisting of a basic holomorphic bundle E and a section θ of $\mathrm{End}E \otimes \bigwedge^{1,0}$ such that:

- $\bar{\partial}_{\mathrm{End}E}\theta = 0$,
- $\theta \wedge \theta = 0$.

Theorem (Biswas-Kasuya'19)

A basic Higgs bundle $(E, \theta) \to M$ over a compact Sasakian manifold admits a basic Hermitian metric such that $\nabla^h + \theta + \theta^{*h}$ is flat if and only if (E, θ) is polystable and $c_{1,B}(E) = c_{2,B}(E) = 0$.

Here, we denote by ∇^h the basic Chern connection of h.

Main Theorem

Let $(M, T^{1,0}, (\xi, \eta))$ be a $2n+1$-dimensional compact Sasakian manifold. Fix a Hodge group G_0 of Hermitian type. Let K be the complexification of the real Lie group K_0.

Definition (basic unformizing bundle)

A *basic uniformizing bundle* is a pair $(P_K, \theta : T^{1,0}M \to P_K \times_K \mathfrak{g}^{-1,1})$ consisting of a basic holomorphic principal K-bundle P_K and a basic holomorphic isomorphism $\theta : T^{1,0}M \to P_K \times_K \mathfrak{g}^{-1,1}$.

Definition (basic uniformizing variations of Hodge structure)

A *basic uniformizing variations of Hodge structure* is a pair $((P_K, \theta), P_{K_0} \subseteq P_K)$ consisting of a basic uniformizing bundle (P_K, θ) and a basic K_0-reduction $P_{K_0} \subseteq P_K$ such that the G_0-connection $D := \nabla + \theta + (-\sigma)(\theta)$ is flat, where ∇ is the canonical connection of the reduction $P_{K_0} \subseteq P_K$, and $\sigma : \mathfrak{g} \to \mathfrak{g}$ is an involution defined as $\sigma(u + \sqrt{-1}v) = u - \sqrt{-1}v$ for $u, v \in \mathfrak{g}_0$.

Let $(M, T^{1,0}, (\xi, \eta))$ be a compact Sasakian manifold. We denote by $\widetilde{M} \to M$ the universal covering, and by $\widetilde{\xi}$ the pull-back of ξ by the covering map. Then the following holds:

Theorem (Kasuya-M.)

The following are equivalent for a bounded symmetric domain $\mathcal{D}$:

(i) There exists a covering map $f : \widetilde{M} \to S^1(\bigwedge^n T^{1,0}\mathcal{D})$ such that $df(\widetilde{\xi}) = 2\pi C \xi_{\mathcal{D}}$ and that $df(T^{1,0}\widetilde{M}) \subseteq \mathbb{C}\xi_{\mathcal{D}} \oplus T^{1,0}(S^1(\bigwedge^n T^{1,0}\mathcal{D}))$ for some positive constant C, where $\xi_{\mathcal{D}}$ denotes the Reeb vector field of $S^1(\bigwedge^n T^{1,0}\mathcal{D})$.

(ii) For some Hodge group G_0 such that $G_0/K_0 = \mathcal{D}$, there exists a uniformizing variations of Hodge structure $(P_K, \theta, P_{K_0} \subseteq P_K)$ such that the following holds for some positive constant C:

$$c_{1,B}(L) = -C[d\eta],$$

where L is a basic complex line bundle defined as $L := P_{K_0} \times_\chi \mathbb{C}$.

Remark

If the above (i) and (ii) hold, then $L \simeq \bigwedge^n T^{1,0}$.

Remark

The complex line bundle L is defined not only for a uniformizing VHS but also for arbitrary G_0-harmonic bundle $(P_{G_0}, D_{G_0}, P_{K_0} \subseteq P_{G_0})$, where D_{G_0} is a flat connection on P_{G_0}. Let $\Phi : \widetilde{M} \to \mathcal{D}$ be the corresponding harmonic map. Then we have $\Phi^* \bigwedge^n T^{1,0}\mathcal{D} = \pi^* L$, where $\pi : \widetilde{M} \to M$ is the projection.

Remark

Obviously, L is trivial if and only if P_{K_0} admits a K_0'-reduction $P_{K_0'} \subseteq P_{K_0}$.

Proof.

Suppose that (i) holds. We regard $\widetilde{M}$ as a Sasakian manifold with a Sasakian stucture induced by the map f. Since $\pi_1(M)$ acts on $\widetilde{M}$ preserving the transverse complex structure, we have a representation $\pi_1(M) \to \mathrm{Aut}(\mathcal{D})$, and a $G_0 := \mathrm{Aut}(\mathcal{D})$-basic uniformizing variations of Hodge structure. Let $\Phi : \widetilde{M} \to \mathcal{D}$ be the corresponding harmonic map. As remarked above, $\Phi^* \bigwedge^n T^{1,0}\mathcal{D} = \pi^* L$ and thus $c_{1,B}(L) = -C[d\eta]$. □

Proof.

Suppose that (ii) holds. Let $\Phi : \widetilde{M} \to \mathcal{D}$ be the harmonic map associated with the uniformizing variations of Hodge structure. Since $\bigwedge^n d\Phi : \bigwedge^n T^{1,0}\widetilde{M} \to \bigwedge^n T^{1,0}\mathcal{D}$ is isomorphic at each fiber, we see $L = \bigwedge^n T^{1,0}$. From the assumption $c_{1,B}(L) = -C[d\eta]$, we see $c_{1,B}(T^{1,0}) = -C[d\eta]$ and thus M is quasi-regular. Since $L = \bigwedge^n T^{1,0}$, the S^1-action lifts to L. From the theory of S^1-equivariant cohomology and the assumption $c_{1,B}(L) = -C[d\eta]$, we can take a global trivialization $s : M \to L$ such that $s(t \cdot x) = t \cdot s(x) e^{2\pi\sqrt{-1}Ct}$ for all $x \in M$ and $t \in \mathbb{R}$, where the left $\mathbb{R}$-action is defined by the Reeb vector field, and the right multiplication of $e^{2\pi\sqrt{-1}Ct}$ is the fiverwise multiplication. This gives a lift $f : \widetilde{M} \to G_0/K_0' = S^1(\bigwedge^n T^{1,0}D)$ of Φ such that $df(\widetilde{\xi}) = 2\pi C\xi_{\mathcal{D}}$. Then by the same argument as the Kähler manifold case, we see that f is a covering map. □

Example

Let $(M, T^{1,0}, (\xi, \eta))$ be a $2n+1$-dimensional compact Sasakian manifold. We define a basic holomorphic vector bundle E as

$$E := T^{1,0}X \oplus \mathbb{C},$$

where $\mathbb{C}$ is the trivial bundle. We also define a basic Higgs field θ on E as

$$\theta := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where 1 is considered to be the identity of $T^{1,0} \otimes (T^{1,0})^* \simeq \mathrm{End}(T^{1,0})$.

Example

(continued from the previous page) Suppose that (E, θ) is stable and that

$$\int_M \{2c_{2,B}(T^{1,0}X) - c_{1,B}(T^{1,0}X)^2\} \wedge (d\eta)^n \wedge \eta = 0.$$

Then the Hermitian-Einstein metric on (E, θ) gives a $PU(n,1)$-basic uniformizing variations of Hodge structure. Suppose also that $c_{1,B}(T^{1,0}) = -C[d\eta]$ for some positive C. Then there exists a covering $f : \widetilde{M} \to S^1(\bigwedge^n T^{1,0}\mathcal{D}) \simeq PU(n,1)/SU(n)$ such that $df(\widetilde{\xi}) = 2\pi\xi_{\mathcal{D}}$ and that $df(T^{1,0}\widetilde{M}) \subseteq \mathbb{C}\xi_{\mathcal{D}} \oplus T^{1,0}(S^1(\bigwedge T^{1,0}\mathcal{D}))$, where we denote by $\mathcal{D}$ the unit ball in $\mathbb{C}^n$.