Multiplier Hermitian-Einstein metrics on KSM manifolds

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X : Fano mfd (i.e., $c_1(X) > 0$), $\dim X = n$

$\omega = \sqrt{-1}\sum_{i,j=1}^{n} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j \in 2\pi c_1(X)$: Kähler form

$\mathrm{Ric}(\omega) := \sqrt{-1}\,\bar{\partial}\partial \log \det(g_{i\bar{j}})$: Ricci form of ω

ω : Einstein-Kähler (EK) $\overset{\mathrm{def}}{\Longleftrightarrow}$ $\mathrm{Ric}(\omega) = \omega$

$^{\exists}\rho_\omega \in C^\infty(X)$: Ricci potential s.t. $\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\,\partial\bar{\partial}\rho_\omega$

& $\int_X e^{\rho_\omega}\omega^n = \int_X \omega^n$

Futaki

$$F_X : H^0(X; \mathcal{O}(TX)) \longrightarrow \mathbb{C}$$: Futaki invariant

$$v \longmapsto \int_X v\rho_\omega\, \omega^n$$

- independent of $\omega \in 2\pi c_1(X)$.
- $^{\exists}$EK on $X \Rightarrow F_X \equiv 0$.

- ω : Kähler-Ricci soliton (KRS) with soliton v.f. V

$\overset{\text{def}}{\Longleftrightarrow}$ V = hol. v.f. on X

$$\mathrm{Ric}(\omega) - \omega = L_V \omega \qquad \left(\begin{array}{l} F_X \equiv 0 \\ \Rightarrow V = 0 \end{array} \right)$$

$\theta_V^{(\omega)} \in C^\infty(X)$ s.t. $i(V)\omega = \sqrt{-1}\,\bar\partial \theta_V^{(\omega)}$ & $\int_X \theta_V^{(\omega)} \omega^n = 0$

$$\mathrm{Ric}(\omega) - \omega = L_V \omega = \sqrt{-1}\,\partial\bar\partial \theta_V^{(\omega)} \quad \therefore \quad P_\omega = \theta_V^{(\omega)} + {}^\exists C$$

- ω : Mabuchi soliton (MS)

$\overset{\text{def}}{\Longleftrightarrow}$ V : hol. v.f. on X

$$e^{P_\omega} - 1 = \theta_V^{(\omega)}$$

$$\left(\begin{array}{l} pr : L^2(X,\omega) \to \mathcal{H}_\omega := \left\{ h \in C^\infty(X) \,\middle|\, \sum_{i,j=1}^n g^{\bar j i} \dfrac{\partial h}{\partial \bar z^j} \dfrac{\partial}{\partial z^i} : \mathrm{hol} \ \& \int_X h\omega^n = 0 \right\} \\ pr(1 - e^{P_\omega}) = pr(S(\omega) - n) \quad (S(\omega) : \text{scalar curv of } \omega) \\ \therefore -V : \text{extremal v.f.} \end{array} \right)$$

V : hol. v. f. on X

$\sigma(t)$: non-const C^∞ ftn on $I=(\alpha,\beta)$

$$-\infty \leqq \alpha \leqq \min_X \theta_V^{(\omega)} \leqq \max_X \theta_V^{(\omega)} \leqq \beta \leqq +\infty$$

$$\dot{\sigma}(t) \leqq 0 \leqq \ddot{\sigma}(t) \quad \text{or} \quad \ddot{\sigma}(t) > 0$$

$\tilde{\omega} := e^{-\sigma(\theta_V^{(\omega)})/n}\omega$: multiplier Hermite-Einstein (mHE)

(ω : σ-soliton w.r.t. V)

$$\overset{\text{def}}{\Longleftrightarrow} \quad Ric_V^\sigma(\omega) := Ric(\omega) + \sqrt{-1}\partial\bar{\partial}\sigma(\theta_V^{(\omega)}) = \omega$$

$$\Longleftrightarrow \quad e^{\rho_\omega} = e^{-\sigma(\theta_V^{(\omega)})}$$

- $\sigma(t) = -t + C \implies$ σ-soliton = KRS
- $\sigma(t) = -\log(t+1) \implies$ σ-soliton = MS

Obstruction (Futaki)

$$F_v^{\sigma}(Z) := \int_X Z\left(\rho_\omega + \sigma(\theta_v^{(\omega)})\right) e^{-\sigma(\theta_v^{(\omega)})}\omega^n \qquad (Z : \text{hol. v.f. on } X)$$

$$= -\int_X \tilde{\theta}_Z^{(\omega)} e^{-\sigma(\theta_v^{(\omega)})}\omega^n$$

$\tilde{\theta}_Z^{(\omega)} \in C^\infty(X)$ s.t.
$i(Z)\omega = \sqrt{-1}\bar{\partial}\tilde{\theta}_Z^{(\omega)}$ & $\int_X \tilde{\theta}_Z^{(\omega)} e^{\rho_\omega}\omega^n = 0$

- independent of $\omega \in 2\pi c_1(X)$.
- $\exists\,\sigma$-soliton $\Longrightarrow F_v^\sigma \equiv 0$

- $\sigma(t) = -t + c$ (KRS)

  $\Rightarrow$ 「$F_v^\sigma \equiv 0 \iff \int_X \tilde{\theta}_Z^{(\omega)} e^{\theta_v^{(\omega)}}\omega^n = 0 \quad (\forall Z)$」

  $\therefore\ \exists! V$ s.t. $F_V^\sigma \equiv 0$ (Tian-Zhu)

- $\sigma(t) = -\log(t+1)$ (MS)

  $\Rightarrow$ 「$F_v^\sigma \equiv 0 \iff \int_X \tilde{\theta}_Z^{(\omega)}(\theta_v^{(\omega)}+1)\omega^n = 0 \quad (\forall Z)$」

  $\iff \int_X \tilde{\theta}_Z^{(\omega)}\theta_v^{(\omega)}\omega^n = \int_X Z(\rho_\omega)\omega^n \quad \therefore\ -V$: extremal v.f. (Futaki-Mabuchi)

  ∵ $\square_\omega \tilde{\theta}_Z^{(\omega)} + \tilde{\theta}_Z^{(\omega)} + Z(\rho_\omega) = 0$

(σ, v)-Ding functional

$\omega_0, \omega \in 2\pi c_1(X)$ $\omega = \omega_0(\varphi) := \omega_0 + \sqrt{-1}\partial\bar\partial\varphi$ $(\varphi \in C^\infty(X))$

$\int_X e^{-\sigma(\theta_v^{(\omega)})}\omega^n$: indep. of $\omega \in 2\pi c_1(X)$; normalization $\int_X e^{-\sigma(\theta_v^{(\omega)})}\omega^n = \int_X \omega_0^n$

$$D_v^\sigma(\varphi) := -\frac{1}{\int_X \omega_0^n}\int_0^1\left\{\int_X \dot\varphi_t e^{-\sigma(\theta_v^{\omega_0(\varphi_t)})}\omega_0(\varphi_t)^n\right\}dt - \log\left(\frac{1}{\int_X\omega_0^n}\int_X e^{\rho_{\omega_0}-\varphi_t}\omega_0^n\right)$$

$$= -\frac{1}{\int_X \omega_0^n}\int_0^1\left\{\dot\varphi_t\left(e^{-\sigma(\theta_v^{(\omega_0(\varphi_t))})} - e^{\rho_{\omega_0(\varphi_t)}}\right)\omega_0(\varphi_t)^n\right\}dt$$

(σ, v)-Ding functional

$\varphi_t \in C^\infty(X)$ $(0 \le t \le 1)$; $\varphi_0 = 0$, $\varphi_1 = \varphi$ $\omega_0(\varphi_t)$: Kähler form

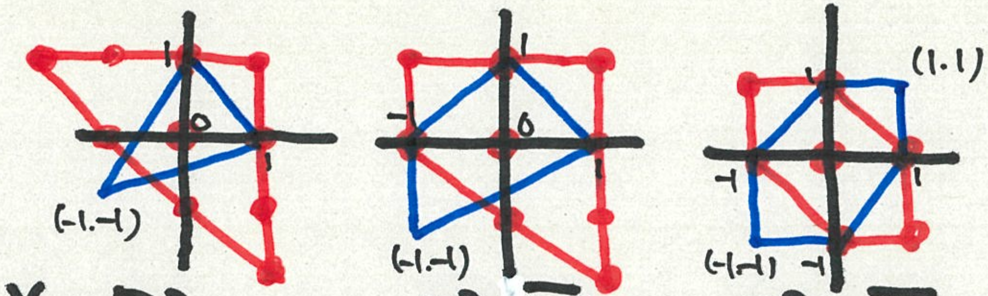
- $D_v^\sigma(\varphi)$ indep. of $\{\varphi_t\}_{0\le t\le 1}$; D_v^σ : well-defined

- $\omega = \omega(\varphi)$: σ-soliton

$\iff$ φ : critical pt of D_v^σ

Toric Fano mfd

e.g. $\ell=2$ P P^*



$X_P : \mathbb{P}^2$ $\mathbb{P}^2 \# \overline{\mathbb{P}^2}$ $\mathbb{P}^2 \# 3\overline{\mathbb{P}^2}$

P : ℓ-dim Fano polytope

$\updownarrow$ 1:1

X_P : toric Fano ℓ-fold

$$P^* := \{ y \in \mathbb{R}^\ell \mid \langle y, x \rangle \leqq 1 \ \ (\forall x \in P) \}$$

for $t = (t_1, \cdots, t_\ell) \in (\mathbb{C}^*)^\ell$, $a = (a_1, \cdots, a_\ell) \in \mathbb{Z}^\ell$, $t^a := t_1^{a_1} t_2^{a_2} \cdots t_\ell^{a_\ell} \in \mathbb{C}^*$

$$\begin{array}{ccc} \varphi_P : (\mathbb{C}^*)^\ell & \hookrightarrow & \mathbb{P}^N(\mathbb{C}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ t & \longmapsto & [t^{-a_0} : t^{a_1} : \cdots : t^{-a_N}] \end{array}$$

inj. hol. $(P^* \cap \mathbb{Z}^\ell = \{a_0, a_1, \cdots, a_N\})$

$$X_P := \overline{\varphi_P((\mathbb{C}^*)^\ell)} \subset \mathbb{P}^N(\mathbb{C})$$

$$\begin{array}{ccc} X_P & \hookrightarrow & \mathbb{P}(H^0(X_P ; K_{X_P}^{-1})^*) \\ \cup_{\text{dense}}^{\text{open}} & & \| \\ (\mathbb{C}^*)^\ell & \xhookrightarrow{\varphi_P} & \mathbb{P}^N(\mathbb{C}) \end{array}$$

Kodaira embedding

Wang-Zhu

Every toric Fano mfd admits a KRS.

Yao (Nakamura)

X_P : toric Fano ℓ-fold

$\exists$ MS on X_P

$\Longleftrightarrow$ $\ell_P > 0$ on P^*

ℓ_P : affine linear ftn on $\mathbb{R}^\ell$ s.t.

$$\int_{P^*} z_k \ell_P(z)\,dz = 0 \quad (k=1,\cdots,\ell) \quad \& \quad \int_{P^*} \ell_P(z)\,dz = 1$$

· $\ell_P > 0$ on P^* $\Longleftrightarrow$ $\theta_V^{(\omega)} > -1$ on X_P (−V: extremal v.f.)

i.e., $\mathrm{Image}(\theta_V^{(\omega)}) \subset (-1, +\infty)$

$v(t) = -\log(t+1)$ defined on $(-1, +\infty)$

KSM-data $(W; L_1, \cdots; L_\ell; P)$

- W : Fano n-fold with EK form ν_0
- for each $i = 1, \cdots, \ell$,
  (L_i, h_i) : holo. Herm line bundle over W
  s.t. eigen values of $\sqrt{-1}\bar\partial\partial \log h_i$ w.r.t. ν_0 : constant $\mu_1^{(i)}, \cdots, \mu_n^{(i)}$
- for each $w \in W$,
  $\exists (U; z^1, \cdots, z^n)$: holo. local coord around w
  $\exists e_i$: holo. local frame for L_i on U
  s.t.
  - $\nu_0 = \sqrt{-1}\sum_{j=1}^n dz^j \wedge d\bar z^j$ at w
  - $h_i(e_i, e_i) = 1$ at w
  - $d(h_i(e_i, e_i)) = 0$ at w
  - $\sqrt{-1}\bar\partial\partial \log(h_i(e_i, e_i)) = \sqrt{-1}\sum_{j=1}^n \mu_j^{(i)} dz^j \wedge d\bar z^j$ at w
  - $\mu_j^{(i)}$ $(j = 1, \cdots, n;\ i = 1, \cdots, \ell)$: const. indep. of $w \in W$
- P : ℓ-dim Fano polytope

$$-\mu_j := -(\mu_j^{(1)}, \mu_j^{(2)}, \cdots, \mu_j^{(\ell)}) \in P^\circ := \mathrm{interior}(P)$$

$$(j = 1, \cdots, n)$$

$(W; L_1, \cdots; L_\ell; P)$: KSM-data

for $\mathbf{a} = (a_1, \cdots, a_\ell) \in \mathbb{Z}^\ell$

$$\mathbb{L}^{\mathbf{a}} := L_1^{\otimes a_1} \otimes L_2^{\otimes a_2} \otimes \cdots \otimes L_\ell^{\otimes a_\ell}$$

$Q (\subset L_1 \oplus \cdots \oplus L_\ell)$: the ass. $(\mathbb{C}^*)^\ell$-bundle over W

for $g = g_1 \oplus \cdots \oplus g_\ell \in Q$

$$(g_i \in L_i \setminus (\text{zero-section}) \ (i = 1, \cdots, \ell))$$

$$g^{\mathbf{a}} := g_1^{\otimes a_1} \otimes g_2^{\otimes a_2} \otimes \cdots \otimes g_\ell^{\otimes a_\ell} \in \mathbb{L}^{\mathbf{a}} \setminus (\text{zero-section})$$

$$E := \bigoplus_{\mathbf{a} \in P^* \cap \mathbb{Z}^\ell} \mathbb{L}^{-\mathbf{a}} = \mathbb{L}^{-\mathbf{a}_0} \oplus \mathbb{L}^{-\mathbf{a}_1} \oplus \cdots \oplus \mathbb{L}^{-\mathbf{a}_N}$$

$$(P^* \cap \mathbb{Z}^\ell = \{\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_N\})$$

$\Phi_P : Q \hookrightarrow \mathbb{P}(E) := (E \setminus (\text{zero-section}))/\mathbb{C}^*$ inj hol

$q \longmapsto [q^{-a_0} \oplus q^{-a_1} \oplus \cdots \oplus q^{-a_N}]$

$Z_0(W; L_1, \cdots ; L_\ell ; P) := \Phi_P(Q) \subset \mathbb{P}(E)$ $(\mathbb{C}^*)^\ell$-bundle over W

$Z(W; L_1, \cdots ; L_\ell ; P) := \overline{Z_0(W; L_1, \cdots ; L_\ell ; P)} \subset \mathbb{P}(E)$ X_P-bundle over W

KSM manifold associated to $(W; L_1, \cdots, L_\ell ; P)$

Example

- $\ell = 1$, $P = [-1, 1]$, $-1 < \mu < 1$; $Z(W; L; [-1, 1])$: $\mathbb{P}^1$-bundle over W

  e.g. $Z(\mathbb{P}^1; \mathcal{O}(1); [-1, 1]) = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}) = \mathbb{P}^2 \# \overline{\mathbb{P}^2}$ $\left(-\mu = \frac{-1}{2} \in (-1, 1)\right)$

- $\ell = 2$, $Z(\mathbb{P}^1; \mathcal{O}(-1), \mathcal{O};$  $) = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O})$ $\left(-\mu = (-\frac{1}{2}, 0) \in P^\circ\right)$

  $(= \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O})$ in algebro-geometric notation$)$ Fano 3-fold

  $Z(\mathbb{P}^1; \mathcal{O}(1), \mathcal{O};$  $) = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O})$ $\left(-\mu = (-\frac{1}{2}, 0) \notin P^\circ\right)$

  NOT Fano

Koiso, Sakane, Mabuchi

$(W; L; [-1,1])$: KSM-data

- $^{\exists}$EK on $Z(W;L;[-1,1])$ ($\mathbb{P}^1$-bundle over W)

$$\Longleftrightarrow \int_{-1}^{1} z \prod_{j=1}^{n} (1+\mu_j z)\, dz = 0$$

- $Z(W;L;[-1,1])$ always admits a KRS.

N.

$(W; L_1, \cdots, L_\ell; P)$: general KSM-data

$Z(W; L_1, \cdots, L_\ell; P)$ always admits a KRS.

In particular,

$^{\exists}$EK on $Z(W; L_1, \cdots, L_\ell; P) \Longleftrightarrow \int_{P^*} z_k \prod_{\alpha=1}^{n} (1+\langle \mu_\alpha, z\rangle)\, dz = 0 \quad (k=1,\cdots,\ell)$

$Z(W; L_1, \cdots, L_\ell; P)$ is Fano

$\because$ · for $i = 1, \cdots, \ell$, $\xi_i \in L_i \setminus$ (zero-section)

$$x_i(\xi_i) := -\log h_i(\xi_i, \xi_i)$$

$$x := (x_1, x_2, \cdots, x_\ell)$$

$\xi_i = t_i e_i$

t_i : fiber coord for L_i

$(i = 1, \cdots, \ell)$

· for $y = (y_1, \cdots, y_\ell) \in \mathbb{R}^\ell$,

$$U_P(y) := \log\Big(\sum_{a \in P^* \cap \mathbb{Z}^\ell} e^{\langle a, y\rangle}\Big) = \log\Big(\sum_{i=0}^{N} e^{\langle a_i, y\rangle}\Big)$$

$$(P^* \cap \mathbb{Z}^\ell = \{a_0, a_1, \cdots, a_N\})$$

$\zeta : Q \longrightarrow W$: proj $((\mathbb{C}^*)^\ell$- bundle$)$

t_i : fiber coord. for L_i $(\xi_i \in L_i,\ \xi_i = t_i(\xi_i) e_i)$

$$\eta_0 := \frac{(n+\ell)!}{n!} e^{-U_P(x)} (\zeta^* \nu_0)^n \wedge \Big(\prod_{i=1}^{\ell} \frac{\sqrt{-1}\, dt_i \wedge d\bar{t}_i}{|t_i|^2}\Big)$$

volume form on Q

Then η_0 extends to the volume form on $Z(W;L_1,\cdots,L_\ell;P)$.

$$\omega_0 := \sqrt{-1}\,\bar\partial\partial\log\eta_0 \in 2\pi\, c_1(Z(W;L_1,\cdots,L_\ell;P))$$

by using the local coord. & local frames in the definition of KSH-data,

$$\omega_0 = \underbrace{\sqrt{-1}\sum_{i,j=1}^{\ell}\frac{\partial^2 u_P}{\partial y_i\partial y_j}(x)\frac{dt_i}{t_i}\wedge\frac{d\bar t_j}{\bar t_j}}$$

(pull-back of Fubini-Study form by φ_P ∴ pos. def.

$$+\sqrt{-1}\sum_{k=1}^{n}\Big(1+\sum_{\alpha=1}^{\ell}\mu_k^{(\alpha)}\frac{\partial u_P}{\partial y_\alpha}(x)\Big)dz^k\wedge d\bar z^k$$

$$m := \Big(\frac{\partial u_P}{\partial y_1}(x),\cdots,\frac{\partial u_P}{\partial y_\ell}(x)\Big): Z(W;L_1,\cdots,L_\ell;P)\longrightarrow\mathbb{R}^\ell$$ moment map

<u>FACT</u> (Mabuchi)

$$m(Z(W;L_1,\cdots,L_\ell;P)) = P^*$$

$\therefore \left(\frac{\partial u_P}{\partial y_1}(x), \cdots, \frac{\partial u_P}{\partial y_\ell}(x)\right) \in P^* = \{ y \in \mathbb{R}^\ell \mid \langle y, x \rangle \leqq 1 \ (\forall x \in P) \}$

$-(\mu_j^{(1)}, \mu_j^{(2)}, \cdots, \mu_j^{(\ell)}) \in P^\circ = \mathrm{interior}(P) \quad (j = 1, \cdots, n)$

$\Longrightarrow \quad 1 + \sum_{\alpha=1}^{\ell} \mu_k^{(\alpha)} \frac{\partial u_P}{\partial y_\alpha}(x) > 0 \quad (k = 1, \cdots, n)$

$\therefore$ ω is a Kähler form on $Z(W; L_1, \cdots, L_\ell; P)$

$\therefore$ $Z(W; L_1, \cdots, L_\ell; P)$ is Fano. //

Example

$$W := \mathbb{P}^{n_1}(\mathbb{C}) \times \mathbb{P}^{n_2}(\mathbb{C}) \times \cdots \times \mathbb{P}^{n_m}(\mathbb{C})$$

$$L_i := \bigotimes_{j=1}^{m} p_j^* \mathcal{O}_{\mathbb{P}^{n_j}}(k_j^{(i)}) \qquad (i=1,\cdots,\ell)$$

$$\left(p_j : W \longrightarrow \mathbb{P}^{n_j}(\mathbb{C}) \quad j\text{-th projection } (j=1,\cdots,m) \right)$$

$$\mu_j^{(i)} := \frac{k_j^{(i)}}{n_j+1} \qquad (j=1,\cdots,m;\ i=1,\cdots,\ell)$$

$$\boldsymbol{\mu}_j := (\mu_j^{(1)}, \mu_j^{(2)}, \cdots, \mu_j^{(\ell)}) \in \mathbb{Q}^\ell \quad (j=1,\cdots,m)$$

multiplicity : n_j

P : ℓ-dim Fano polytope s.t. $-\boldsymbol{\mu}_j \in P^\circ$ $(j=1,\cdots,m)$

$\Longrightarrow$ $(W; L_1,\cdots,L_\ell; P)$: KSM-data

$Z(W; L_1,\cdots,L_\ell; P)$: トーリック Fano mfd

$(W; L_1, \cdots, L_\ell; P)$: KSM-data

$q = q_1 \oplus \cdots \oplus q_\ell \in Q \ (\subset L_1 \oplus \cdots \oplus L_\ell)$

$x_i(q) := -\log h_i(q_i, q_i) \quad (i = 1, \cdots, \ell)$,

$x := (x_1, \cdots, x_\ell)$

$T_\ell := (\mathbb{C}^*)^\ell \supset S_\ell := (S^1)^\ell$

$\varphi \in C^\infty(X_P)^{S_\ell}$ is a ftn of $y = (y_1, \cdots, y_\ell)$

($y_i := -\log|t_i|^2 \ (i = 1, \cdots, \ell;\ t = (t_1, \cdots, t_\ell) \in T_\ell)$)

$\varphi(x)$ can be regrded as a ftn on $Z(W; L_1, \cdots, L_\ell; P)$

$\mathcal{F}_1$: the set of such ftns on $Z(W; L_1, \cdots, L_\ell; P)$

$$\eta_0 := \frac{(n+\ell)!}{n!} e^{-u_P(x)} (\zeta^* \nu_0)^n \wedge \left(\prod_{i=1}^{\ell} \frac{\sqrt{-1}\, dt_i \wedge d\bar{t}_i}{|t_i|^2} \right)$$

$\omega_0 := \sqrt{-1}\, \bar\partial \partial \log \eta_0 \ \in 2\pi c_1(Z(W; L_1, \cdots, L_\ell; P))$

$\varphi \in \mathcal{F}_1$, $\eta := e^{-\varphi}\eta_0$, $u := u_P + \varphi$

$$\omega := \sqrt{-1}\,\bar{\partial}\partial \log \eta = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi$$

Assumption

V : fiber-directed v.f. on $Z(W; L_1, \cdots ; L_\ell ; P)$

$$\left(V := \sum_{i=1}^{\ell} c_i t_i \frac{\partial}{\partial t_i} \qquad (c_1, \cdots, c_\ell) \in \mathbb{R}^\ell \right)$$

$$\Rightarrow \theta_V^{(\omega)}(x) = -\sum_{i=1}^{\ell} c_i \frac{\partial u}{\partial y_i}(x) + {}^{\exists}C_V$$

- $F_V^\sigma \equiv 0 \iff \displaystyle\int_{P^*} z_k \prod_{\alpha=1}^{n} (1 + \langle \mu_\alpha, z \rangle) e^{-\sigma(-\langle c, z\rangle + C_V)} dz = 0$

$(k = 1, \cdots, \ell)$

- ω : σ-soliton

$$\iff \det\left(\frac{\partial^2 u}{\partial y_i \partial y_j}(x)\right) \prod_{\alpha=1}^{n}\left(1 + \sum_{k=1}^{\ell} \mu_\alpha^{(k)} \frac{\partial u}{\partial y_k}(x)\right) = e^{-u(x) + \sigma(\theta_V^{(\omega)}(x))}$$

$$g(\mathbb{z}) := \prod_{\alpha=1}^{n} (1 + \langle \mu_\alpha . \mathbb{z} \rangle) e^{-\sigma(-\langle \mathbb{c}, \mathbb{z} \rangle + C_V)}$$ fcn on $\mathbb{R}^l$

$$F_V^\sigma \equiv 0 \iff \int_{P^*} z_k g(\mathbb{z}) d\mathbb{z} = 0 \quad (k = 1, \cdots, l)$$

$$\omega : \sigma\text{-soliton} \iff g(\nabla u(y)) \det(\nabla^2 u(y)) = e^{-u(y)}$$

Thm A (N. - Nakamura)

Assume V : fiber-directed

$\exists \omega$: σ-soliton w.r.t. V

$$\iff \int_{P^*} z_k g(\mathbb{z}) d\mathbb{z} = 0 \quad (k = 1, \cdots, l)$$

Berman - Berndtsson

Δ : convex body ($\subset \mathbb{R}^\ell$), $\mathbf{0} \in \Delta^\circ$ (= interior of Δ)

$g(\mathbf{z})$: positive smooth ftn on Δ

$$\int_\Delta z_k\, g(\mathbf{z})\, d\mathbf{z} = 0 \quad (k = 1, \cdots, \ell)$$

(i.e., barycenter of Δ w.r.t. $g(\mathbf{z})d\mathbf{z}$ is $\mathbf{0}$)

$\Rightarrow \exists \phi$: smooth convex ftn on $\mathbb{R}^\ell$

s.t. $$\begin{cases} g(\nabla\phi)\det(\nabla^2\phi) = e^{-\phi} \\ \nabla\phi = \left(\dfrac{\partial\phi}{\partial z_1}, \cdots, \dfrac{\partial\phi}{\partial z_\ell}\right) : \mathbb{R}^\ell \to \Delta^\circ \ \text{: diffeo} \end{cases}$$

$+$

Saito - Takahashi (multiplier Hermitian Ricci flow)

$$PSH(P^*) := \{ u : \mathbb{R}^\ell \to \mathbb{R} \text{ convex} \mid u \leqq v_{P^*} + {}^{\exists}C \}$$

$$v_{P^*}(y) := \sup_{x \in P^*} \langle y, x \rangle$$

$$\mathcal{E}^1(P^*) := \{ u \in PSH(P^*) \mid \int_{P^*} u^*(x)\, dx < +\infty \}$$

$$u^*(x) := \sup_{y \in \mathbb{R}^\ell} (\langle y, x \rangle - u(y)) \quad : \text{Legendre dual of } u$$

(σ, v)-Ding functional D_v^σ on $\mathcal{E}^1(P^*)$ is reduced to

$$D_v^\sigma(u) := \frac{1}{|P^*|_g} \int_{P^*} u^*(x)\, g(x)\, dx - \log \int_{\mathbb{R}^\ell} e^{-u(y)}\, dy \qquad (u \in \mathcal{E}^1(P^*))$$

$$\left(|P^*|_g := \int_{P^*} g(x)\, dx \right)$$

$$h(x) := \prod_{\alpha=1}^{n} (1 + \langle \mu_\alpha, x \rangle) \qquad \left(g(x) = h(x) e^{-\sigma(-\langle \xi, x \rangle + C_v)} \right)$$

D_V^σ : coercive on $\mathcal{E}^1(P^*)$

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists\, \delta, C > 0$ s.t. $D_V^\sigma(u) \geqq \delta J_{\text{red}}(u) - C \quad (\forall u \in \mathcal{E}^1(P^*))$

$$J_{\text{red}}(u) := \inf\left\{ \frac{1}{|P^*|_h} \int_{P^*} (u^*(z) - \ell(z))\, h(z)\, dz - \inf_{P^*} (u^* - \ell) \;\middle|\; \ell : \text{affine fcn on } P^* \right\}$$

$$\left(|P^*|_h := \int_{P^*} h(z)\, dz \right)$$

<u>Thm B</u> (N.-Nakamura)

Assume V : fiber-directed

D_V^σ : coercive on $\mathcal{E}^1(P^*)$

$$\Longleftrightarrow \quad \int_{P^*} z_k\, g(z)\, dz = 0 \quad (k = 1, \cdots, \ell)$$

$$e^{-\sigma(-\langle a, z\rangle + c(v))} > 0 \quad \text{on } P^*$$

$$h(z) > 0 \quad \text{on } P^*$$

<u>Lemma</u> $\exists C > 0$ s.t.

$$J_{red}(u) \leqq \frac{1}{|P^*|_h}\int_{P^*}\left(u^*(z) - \langle a, z\rangle - u^*(0)\right)h(z)\,dz \leqq C\,J_{red}(u)$$

for $\forall u \in \mathcal{E}^1(P^*)$, $a \in \partial u^*(0) := \{a \in \mathbb{R}^l \mid u^*(z) \geqq \langle a, z\rangle + u^*(0) \text{ on } P^*\}$

$(\neq \emptyset \because u^* : \text{convex})$

$\tilde{u}^*(z) := u^*(z) - \langle a, z\rangle - u^*(0) \quad (a \in \partial u^*(0))$

$\Rightarrow \quad \tilde{u}^*(z) \geqq \tilde{u}^*(0) = 0$

$\ell(z) = \langle b, z\rangle + c$: affine ftn $\Rightarrow$ $(u^* + \ell)^*(y) = u(y - a) + \overset{\ell(0)}{\overset{\|}{c}}$

$\therefore \quad D_v^\sigma((u^* + \ell)^*) = D_v^\sigma(u) \quad \left(\because \int_{P^*} z_k\, g(z)\,dz = 0 \ (k = 1, \cdots, l)\right)$

(Yao, Nakamura) $\exists \delta, C > 0$ s.t. $D_v^\sigma((\tilde{u}^*)^*) \geqq \delta \frac{1}{|P^*|_h}\int_{P^*}\tilde{u}^*(z)h(z)\,dz - C$ //

$\lambda_1, \cdots, \lambda_p$: affine ftn with $\mathbb{Q}$-coefficient

$\phi := \max\{\lambda_1, \cdots, \lambda_p\}$: rational PL convex ftn

R : large integer

$$\left(\begin{array}{l} \square := \{ (\mathbf{z}, z_{\ell+1}) \in \mathbb{R}^\ell \oplus \mathbb{R} \mid \mathbf{z} \in P^*,\ 0 \leqq z_{\ell+1} \leqq R - \phi(\mathbf{z}) \} \\ \rightsquigarrow \text{(toric) test configuration for } Z(W; L_1, \cdots, L_\ell; P) \end{array} \right)$$

$u_t := (u_P^* + t(\phi - R))^*$: toric geodesic in $\mathcal{E}^1(P^*)$

$$\Rightarrow \lim_{t \to +\infty} \frac{1}{t} D_v^\sigma(u_t) = \frac{1}{|P^*|_g} \int_{P^*} \phi(\mathbf{z}) g(\mathbf{z}) d\mathbf{z} - \phi(\mathbf{0})$$

$$\mathcal{D}_v^\sigma(\phi) := \frac{1}{|P^*|_g} \int_{P^*} \phi(\mathbf{z}) g(\mathbf{z}) d\mathbf{z} - \phi(\mathbf{0})$$ (σ, v)-Ding invariant

V : fiber-directed

$Z(W; L_1, \cdots, L_\ell; P)$: fiber-directed uniformly relative D-stable

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists \lambda > 0$ s.t

$$\mathcal{D}_V^{\nabla}(\phi) \geqq \lambda J_{red}(\phi^*) \quad \text{for } \forall \phi : \text{rational PL function}$$

<u>Thm C</u> (N.-Nakamura)

Assume V : fiber-directed

$Z(W; L_1, \cdots, L_\ell; P)$: fiber-directed uniformly relative D-stable

$$\Longleftrightarrow \quad \int_{P^*} z_k \, g(z) \, dz = 0 \quad (k = 1, \cdots, \ell)$$

Example

$\sigma_0(t) = -t \quad (-\infty < t < +\infty)$ (KRS)

$\sigma_1(t) = -\log(t+1) \quad (-1 < t < +\infty)$ (MS)

$0 \leqq \tau \leqq 1$,

$$\sigma_\tau(t) := (1-\tau)\sigma_0(t) + \tau\sigma_1(t) = -(1-\tau)t - \tau\log(t+1) \quad (-1 < t < +\infty)$$

$$Z := Z(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(1); [-1, 1]) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$$

$\Rightarrow$ $0 \leqq {}^\forall\tau \leqq 1$,

$\exists\, \sigma_\tau$-soliton on Z

Existence & Uniqueness of V

$\sigma(t)$: non-const ftn on $I=(\alpha,\beta)$ $\quad (-\infty \leqq \alpha \leqq 0 \leqq \beta \leqq +\infty)$

$$\dot{\sigma}(t) \leqq 0 \leqq \ddot{\sigma}(t)$$

$$\left(\int_X \tilde{\theta}_V^{\,\omega} e^{\rho_\omega} \omega^n = 0 \;\Rightarrow\; \mathrm{Image}(\tilde{\theta}_V^{\,\omega}) \ni 0 \right)$$

$\exists t_0$ s.t. $\dot{\sigma}(t_0) < 0$

$$\sigma(t) \geqq \dot{\sigma}(t_0)(t-t_0) + \sigma(t_0)$$

$$\left(e^{-\sigma(t)}\right)' = -\dot{\sigma}(t) e^{-\sigma(t)} \geqq 0$$

Assume: $\exists h(t)$: C^1 ftn on $\mathbb{R}$

s.t.
- $h(t) = e^{-\sigma(t)}$ on I
- $h'(t) \geqq 0$ on $\mathbb{R}$
- $h(t) \leqq e^{-\dot{\sigma}(t_0)(t-t_0) - \sigma(t_0)}$ on $\mathbb{R}$
- $h(t)$: Non-const. on $\forall$ nbd of 0

$$\left(\begin{array}{lll} \text{e.g.} & \sigma(t) = -\log(t+1) & \text{on } (-1, +\infty) \\ & h(t) = t+1 & \text{on } \mathbb{R} \end{array} \right)$$

$$H(t) := \int_{t_0}^{t} h(s)\,ds$$

$$H'(t) = h(t), \quad H''(t) = h'(t) \geqq 0 \quad \text{on } \mathbb{R}$$

$$A := H'(t_0) = h(t_0) = e^{-\sigma(t_0)} > 0$$

$$H : \text{convex} \Rightarrow H(t) \geqq A(t - t_0) + H(t_0) \quad \text{on } \mathbb{R}$$

$$\therefore \lim_{t \to +\infty} H(t) = +\infty$$

for $t \leqq t_0$

$$-H(t) = \int_{t}^{t_0} h(s)\,ds \leqq \int_{t}^{t_0} e^{-\dot{\sigma}(t_0)(s - t_0) - \sigma(t_0)}\,ds$$

$$= -\frac{1}{\dot{\sigma}(t_0)} e^{-\sigma(t_0)} + \frac{1}{\dot{\sigma}(t_0)} e^{-\dot{\sigma}(t_0)(t - t_0) - \sigma(t_0)}$$

$$\therefore H(t) \geqq \frac{1}{\dot{\sigma}(t_0)} e^{-\sigma(t_0)} \quad \text{on } \mathbb{R}$$

$$H_0(t) := H(t) - \frac{1}{\dot{\sigma}(t_0)} e^{-\sigma(t_0)}$$

$$\Rightarrow \begin{cases} \cdot\ H_0(t) \geqq 0 \quad \text{on } \mathbb{R} \\ \cdot\ H_0'(t) = h(t) \\ \cdot\ \lim_{t\to+\infty} H_0(t) = +\infty \end{cases}$$

$V_1, \cdots, V_\ell$: basis for the sp of hol. v.f.'s

$a = (a_1, \cdots, a_\ell) \in \mathbb{R}^\ell$

$$f(a) := \int_X H_0\Big(\sum_{i=1}^{\ell} a_i \tilde{\theta}_{V_i}^{(\omega)}\Big)\omega^n$$

$$\Rightarrow \frac{\partial f}{\partial a_i}(a) = \int_X \tilde{\theta}_{V_i}^{(\omega)} h\Big(\sum_{i=1}^{\ell} a_i \tilde{\theta}_{V_i}^{(\omega)}\Big)\omega^n$$

$$\left(\text{If } \mathrm{Image}\Big(\sum_{i=1}^{\ell} a_i \tilde{\theta}_{V_i}^{(\omega)}\Big) \subset I,\ \frac{\partial f}{\partial a_i} = \int_X \tilde{\theta}_{V_i}^{(\omega)} e^{-\sigma\left(\sum_{i=1}^{\ell} a_i \hat{\theta}_{V_i}^{(\omega)}\right)}\omega^n \right)$$

$$\left(\frac{\partial^2 f}{\partial a_i \partial a_j}(a)\right) = \left(\int_M \tilde{\theta}_{V_i}^{(\omega)} \tilde{\theta}_{V_j}^{(\omega)} h'\Big(\sum_{i=1}^{\ell} a_i \hat{\theta}_{V_i}^{(\omega)}\Big)\omega^n\right) > 0$$

$H_0(t) \geqq 0$ & $\lim_{t \to +\infty} H(t) = +\infty$

(Tian-Zhu) $\Rightarrow$ $f(a)$: proper on $\mathbb{R}^l$

$\therefore$ $\exists^1 a_0 \in \mathbb{R}^l$ s.t. $\dfrac{\partial f}{\partial a_i}(a_0) = 0 \quad (i = 1, \cdots, l)$

$a_0 = (a_1^0, \cdots, a_l^0)$,

$\mathrm{Image}\left(\sum_{i=1}^{l} a_i^0 \tilde{\theta}_{v_i}^{(\omega)}\right) \subset I$

$$\Rightarrow \int_M \tilde{\theta}_{v_i}^{(\omega)} e^{-\sigma\left(\sum_{i=1}^{l} a_i^0 \tilde{\theta}_{v_i}^{(\omega)}\right)} \omega^n = 0 \quad (i = 1, \cdots, l)$$

If we change the def of σ-soliton by

$$\mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}\left(\sigma(\tilde{\theta}_v^{(\omega)})\right) = \omega,$$

then $\exists^1 V$

for $\sigma(t)$ and $c \in \mathbb{R}$, $\sigma_c(t) := \sigma(t+c)$

$$\tilde{\theta}_v^{(\omega)} = \theta_v^{(\omega)} + c_v$$

$$\Rightarrow c_v = \frac{\int_X \tilde{\theta}_v^{(\omega)} \omega^n}{\int_X \omega^n} = \frac{-1}{\int_X \omega^n} \int_X V(\rho_\omega) \omega^n$$ Futaki inv. (indep. of ω)

$$0 = \int_X \tilde{\theta}_z^{(\omega)} e^{-\sigma(\tilde{\theta}_v^{(\omega)})} \omega^n = \int_X \tilde{\theta}_z^{(\omega)} e^{-\sigma_{c_v}(\theta_v^{(\omega)})} \omega^n$$