Analytic Ax-Schanuel Theorem for semi-abelian varieties and Nevanlinna theory

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Folklore of Math.: e **and π are alg. indep.**

$\mathrm{e}^{\pi \mathrm{i}} + 1 = 0$: Transcendental relation.

Do you mind this or not?

1. Ax-Schanuel

Schanuel Conj. (Lang's monog. 1966). Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ be linearly indep. over $\mathbf{Q}$. Then

$$\mathrm{tr.\,deg}_{\mathbf{Q}}\{\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\} \geq n.$$

(i) $n = 1$: Gel'fond-Schneider (1934; Hilbert's 7th Problem).

(ii) $n > 1$: Open. Even in $n = 2$: With $(\alpha_1, \alpha_2) = (1, \pi i)$ it implies the Folklore: alg. indep, of e and π.

(iii) e, e^{π} are alg. indep. (Nesterenko, 1996). The elliptic modular function $j(\tau)$ was used.

Formal Functional Analogue: J. Ax (1971, '72) proved the analogue:

Thm. 1.1 (**Ax-Schanuel**). Let $f(t) = (f_j(t)) \in (\mathbf{C}[[t]])^n$. If $f_j(t) - f_j(0)$, $1 \leq j \leq n$, are linearly independent over $\mathbf{Q}$, then

$$\mathrm{tr.\,deg}_{\mathbf{C}}\{f_1(t), \ldots, f_n(t), e^{f_1(t)}, \ldots, e^{f_n(t)}\} \geq n + 1.$$

More generally, he proved it for semi-ableian varieties, and dealt with t of several variables.

Ax's proof : By means of Kolchin's theory of differential algebra.
（西岡久美子著「微分体の理論」共立.）

Our Aim : 1) Prove Ax-Schanule for entire $f_j(z)$ and a semi-abelian variety A by means of Nevanlina theory,

2) Study and prove a 2nd Main Theorem for the "extended exponential map"

$$\widehat{\exp}_A f : z \in \mathbf{C} \to (\exp_A f(z), f(z)) \in A \times \mathrm{Lie}(A).$$

N.B. There is no "value" in formal analytic functions, but there is for analytic functions: an advantage in the sense that we can think of more problems.

We mainly follow the developments of the theory for entire curves into A since Bloch-Ochiai's Theorem and S. Lang's monog. '66.

Arithmetic Thry.–O-minimal Thry.–Nevnalinna Thry.:

(i) **Raynaud's Theorem** (1983, Manin-Munford Conj.):
$X \subset A$ subvariety $(/K)$. $\Rightarrow X_{\mathrm{tor}} = \bigcup_{\mathrm{finite}}(a + B_{\mathrm{tor}})$,
where $a \in X_{\mathrm{tor}}$ and alg. subgrp's. B.
Proof: By method of char. $p > 0$.

(ii) Another Proof by O-minimal due to Pila-Zannier (2008).

(iii) Yet Another Proof by Nevanliina thry. (Log Bloch–Ochiai)+O-minimal (N., Atti Accad. Naz. Rend. Lincei Mat. Appl. **29** (2018)).

(iv) Another Proof of Ax-Schanuel by "O-minimal" (Tsimerman 2015, Peterzil-Starchenko 2018).

(v) Yet Another Proof of Analy. Ax-Schanuel by Nevanlinna thry. : Today 1.
More on the Value Distribution: Today 2.

······· without "O-minimal".

(vi) **Expectation**: Analy. Ax-Schanuel + O-minimal + Arithmetic $\Longrightarrow$??

Application of Ax-Shanuel:(e.g.) W.D. Brownawell and K.K. Kubota, The algebraic independence of Weierstrass functions and some related numbers, Acta Arith. **33** (1977), 111–149.

This is covered by the present result.

2. Results

Jet Spaces. Let A be a semi-abelian variey of dim n:

$$0 \to (\mathbf{C}^*)^{\mathsf{t}} \to \mathsf{A} \to \mathsf{A}_0 \to 0 \quad \text{(with } \mathsf{A}_0 \text{ abelian var.)},$$

$\exp_{\mathsf{A}} : \mathrm{Lie}(\mathsf{A}) \to \mathsf{A}$ be an exponential map;
$\mathsf{f} : \mathbf{C} \to \mathrm{Lie}(\mathsf{A}) \cong \mathbf{C}^{\mathsf{n}}$ be an entire curve.

Set

$$\widehat{\exp}_{\mathsf{A}}\mathsf{f} : \mathsf{z} \in \mathbf{C} \to (\exp_{\mathsf{A}} \mathsf{f}(\mathsf{z}), \mathsf{f}(\mathsf{z})) \in \mathsf{A} \times \mathrm{Lie}(\mathsf{A}).$$

Take its k-jet lift:

$$\mathsf{J}_{\mathsf{k}}(\widehat{\exp}_{\mathsf{A}}\mathsf{f}) : \mathsf{z} \in \mathbf{C} \to (\mathsf{J}_{\mathsf{k}}(\exp_{\mathsf{A}} \mathsf{f}(\mathsf{z})), \mathsf{J}_{\mathsf{k}}(\mathsf{f}(\mathsf{z}))) \in \mathsf{J}_{\mathsf{k}}(\mathsf{A} \times \mathrm{Lie}(\mathsf{A})) \cong \mathsf{J}_{\mathsf{k}}(\mathsf{A}) \times \mathsf{J}_{\mathsf{k}}(\mathrm{Lie}(\mathsf{A})).$$

Speciality:

$$\begin{array}{cc} \mathsf{J}_{\mathsf{k}}(\mathsf{A}) \cong \mathsf{A} \times \mathsf{J}_{\mathsf{k},\mathsf{A}}, & \mathsf{J}_{\mathsf{k}}(\mathrm{Lie}(\mathsf{A})) \cong \mathrm{Lie}(\mathsf{A}) \times \mathsf{J}_{\mathsf{k},\mathrm{Lie}(\mathsf{A})} \\ \mathsf{J}_{\mathsf{k}}(\mathsf{A} \times \mathrm{Lie}(\mathsf{A})) = \mathsf{A} \times \mathsf{J}_{\mathsf{k},\mathsf{A}} \times \mathrm{Lie}(\mathsf{A}) \times \mathsf{J}_{\mathsf{k},\mathrm{Lie}(\mathsf{A})}, & \mathsf{J}_{\mathsf{k},\mathsf{A}} = \mathsf{J}_{\mathsf{k},\mathrm{Lie}(\mathsf{A})}, \\ \mathsf{J}_{\mathsf{k}}(\widehat{\exp}_{\mathsf{A}}\mathsf{f})(\mathsf{z}) = (\exp_{\mathsf{A}} \mathsf{f}(\mathsf{z}), \mathsf{J}_{\mathsf{k},\exp_{\mathsf{A}} \mathsf{f}}(\mathsf{z}), \mathsf{f}(\mathsf{z}), \mathsf{J}_{\mathsf{k},\mathsf{f}}(\mathsf{z})), & \mathsf{J}_{\mathsf{k},\exp_{\mathsf{A}} \mathsf{f}}(\mathsf{z}) = \mathsf{J}_{\mathsf{k},\mathsf{f}}(\mathsf{z}) \ (\mathsf{k} \geq 1). \end{array}$$

We consider:

$$J_k(\widehat{\exp}_A f)(z) \in A \times \mathrm{Lie}(A) \times J_{k,A} \hookrightarrow J_k(A \times \mathrm{Lie}(A)). \tag{2.1}$$

$\widehat{J}_{k,A} = \mathrm{Lie}(A) \times J_{k,A} \cong \mathbf{C}^n \times \mathbf{C}^{nk}$ is called the **extended jet part**.
$X_k(\widehat{\exp}_A f) = \overline{J_k(\widehat{\exp}_A f)(\mathbf{C})}^{\mathrm{Zar}}$ is the Zariski closure of the image:

$$\mathrm{tr.deg}_{\mathbf{C}}\, \widehat{\exp}_A f := \dim_{\mathbf{C}} X_0(\widehat{\exp}_A f).$$

Def. 2.2. $f : \mathbf{C} \to \mathrm{Lie}(A)$ is <u>A-degnerate</u> if $\exists$ alg. subgroup $G \subsetneqq A$ s.t. $\exp_A f(\mathbf{C}) \subset \exp_A f(0) + G$ (coset type).

N.B. $A = (\mathbf{C}^*)^t$: $f = (f_j)$ is $(\mathbf{C}^*)^n$-degenerate $\iff f_j, 1 \le j \le n$, are lin. dep./$\mathbf{Q}$.

Thm. 2.1 (Analy. Ax-Schanuel). If an entire curve $f : \mathbf{C} \to \mathrm{Lie}(A)$ is A-nondeg., then $\mathrm{tr.deg}_{\mathbf{C}}\, \widehat{\exp}_A f \ge n + 1$.

Order Functions.

$f = (f_1, \ldots, f_n) : z \in \mathbf{C} \to f(z) \in \mathbf{C}^n \cong \mathrm{Lie}(A)$, an entire curve.
Nevanlinna-Shimizu-Ahlfors order function:

$$T(r, f_j) = T_{f_j}(r, \omega_{\mathrm{FS}}) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* \omega_{\mathrm{FS}}.$$

Roughly, <u>$T(r, f_j) \sim \log \max_{|z|=r} |f_j(z)|$</u>.
$T_f(r) := \max_{1 \le j \le n} T(r, f_j)$.

$T_{\exp_A f}(r) = T_{\exp_A f}(r, \omega_L)$ with the curvature form ω_L of a big l.b. $L \to \bar{A}$.
$T_{\widehat{\exp}_A f}(r) := T_{\exp_A f}(r) + T_f(r)$ for $\widehat{\exp}_A f : \mathbf{C} \to A \times \mathrm{Lie}(A)$.
$S_{\exp_A f}(r) = O\left(\log^+ T_{\exp_A f}(r)\right) + O(\log r) + O(1)|| = o(T_{\exp_A f}(r))||$ (with except'l intervals of total finite length).

Lem. 2.3 (Key). (i) $T_f(r) = S_{\exp_A f}(r)$.
(ii) $T_{\widehat{\exp}_A f}(r) = T_{\exp_A f}(r) + S_{\exp_A f}(r)$.

Proof. Use the complex Poisson integral + Borel's technic.

Proof of Analytic Ax-Schanuel Thm.2.1.

The A-nondegeneracy and the Log Bloch–Ochiai imply $\overline{\exp_A f(\mathbf{C})}^{\mathrm{Zar}} = A$:

$$\text{tr.}\deg_{\mathbf{C}} \exp_A f = n. \tag{2.4}$$

Lem. 2.5. $\text{tr.}\deg_{\mathbf{C}(f)} \mathbf{C}(f, \exp_A(f)^* \mathbf{C}(A)) \geq 1$.

Pf. If "$= 0$", $(\exp_A f)^* \mathbf{C}(A)$ is alg. over (f_j), so that $T_{\exp_A f}(r) = O(T_f(r)) = o(T_{\exp_A f}(r))$ by Key Lem. 2.3; Contradiction! △

(2.4) $\Rightarrow$ $\text{tr.}\deg_{\mathbf{C}} \widehat{\exp}_A f \geq n$. Suppose $\text{tr.}\deg_{\mathbf{C}} \widehat{\exp}_A f = n$. $\Rightarrow$ <u>f_j are alg. $/(\exp_A f)^* \mathbf{C}(A)$</u>.
$\Longrightarrow$ $\exists$ non-trivial alg. relations

$$P_j(f_j, \hat{\phi}) = P_j(f_j, \hat{\phi}_1, \ldots, \hat{\phi}_n) = 0, \quad 1 \leq j \leq n, \tag{2.6}$$

where $\{\phi_j\}_{j=1}^n$ is a transcendental basis of $\mathbf{C}(A)$, and $\hat{\phi}_j := \phi_j \circ \exp_A f$.

Lem. 2.5 $\Rightarrow$ $\mathrm{tr.deg}_{\mathbf{C}}\{f_j\}_{j=1}^n < n$: That is, $\exists$ a non-trivial alg. relation

$$Q(f_1, \ldots, f_n) = 0. \tag{2.7}$$

Eliminate f_j $(1 \leq j \leq n)$ in (2.6) and (2.7). $\Rightarrow$ f is A-degnerate: Contradiction! □

Example. (Brownawell-Kubota) A product of elliptic curves, $A := \prod^n E_j$ and alg. indep. $f = (f_j) : \mathbf{C} \to \mathrm{Lie}(A)$:

$$\mathrm{tr.deg}_{\mathbf{C}}\{f_1, \ldots, f_n, \wp_1(f_1), \ldots, \wp_m(f_n)\} \geq n + 1.$$

Here one may claim the same for more generally A-nondegenerate $f = (f_j)$: e.g., with $f_1(z) = z, f_2(z) = z$ and non-isogenious E_j $(j = 1, 2)$,

$$\mathrm{tr.deg}_{\mathbf{C}}\{z, \wp_1(z), \wp_2(z)\} = 3.$$

$\overline{\mathrm{Lie}(E_1) \times \mathrm{Lie}(E_2)} \times A = \mathbf{P}^2(\mathbf{C}) \times E_1 \times E_2,$

$$T_{\widehat{\exp}_A f}(r) = \frac{\pi r^2}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + o(1)\right),$$

wherer λ_j are the areas of the fundamental parallelograms of $\wp_j$ $(j = 1, 2)$.
Let $P(z_1, z_2, w_1, w_2)$ be a polynomials of degrees d_1, d_2 in w_1, w_2 respectively, and $\Xi_P = \{P(z, z, \wp_1(z), \wp_2(z)) = 0\}$. Then

$$N_\infty(r, \Xi_P) = N_1(r, \Xi_P) + o(r^2) = \pi r^2 \left(\frac{d_1}{\lambda_1} + \frac{d_2}{\lambda_2} + o(1)\right).$$

3. Nevanlinna thry. for $\widehat{\exp}_A f$

Thm. 3.1 (2nd Main Thm.). Let $f : \mathbf{C} \to \mathrm{Lie}(A)$ be A-nondegenerate.

(i) For a reduced alg. subset $Z \subset X_k(\widehat{\exp}_A f)$ $(\subset A \times \widehat{J}_{k,A})$ $(k \geqq 0)$, $\exists \bar{A} \times \bar{\widehat{J}}_{k,A}$, a proj. compactification with closures $\bar{X}_k(\widehat{\exp}_A f)$ and $\bar{Z}$ ksuch that

$$T_{J_k(\widehat{\exp}_A f)}(r, \omega_{\bar{Z}}) = N_1(r, J_k(\widehat{\exp}_A f)^* Z) + S_{\varepsilon, \exp_A f}(r), \tag{3.1}$$

where $S_{\varepsilon, \exp_A f}(r) \leq \varepsilon T_{\exp_A f}(r) + O(\log r) \; ||_\varepsilon \; (\forall \varepsilon > 0)$, and $\omega_{\bar{Z}}$ is a sort of curvature form associated with $\bar{Z}$.

(ii) If $\mathrm{codim}_{X_k(\widehat{\exp}_A f)} Z \geqq 2$, then

$$T_{\widehat{\exp}_A f}(r, \omega_{\bar{Z}}) = S_{\varepsilon, \exp_A f}(r). \tag{3.2}$$

(iii) $(k = 0)$ If D is a reduced divisor on $A \times \mathrm{Lie}(A)$ and $D \not\supset X_0(\widehat{\exp}_A f)$, then

$$T_{\widehat{\exp}_A f}(r, \omega_{\bar{D}}) = N_1(r, (\widehat{\exp}_A f)^* D) + S_{\varepsilon, \widehat{\exp}_A f}(r). \tag{3.3}$$

where $\bar{D} \subset \bar{A} \times \overline{\mathrm{Lie}(A)}$.

Pf. $\exists \ell \in \mathbf{N}$ such that

$$T_{J_k(\widehat{\exp}_A f)}(r, \omega_{\bar{Z}}) = N_\ell(r, J_k(\widehat{\exp}_A f)^* Z) + S_{\exp_A f}(r)$$

Here, using this and codim $Z \geq 2$, we prove (ii).

Using (ii), we deduce

$$N_\ell(r, J_k(\widehat{\exp}_A f)^* Z) - N_1(r, J_k(\widehat{\exp}_A f)^* Z) = S_{\varepsilon, \exp_A f}(r),$$

$\Longrightarrow$ (i). □

As an aplication we have:

Thm. 3.2. Let $\widehat{\exp}_A f : \mathbf{C} \to A \times \mathrm{Lie}(A)$ and $\bar{D} \subset \bar{A} \times \overline{\mathrm{Lie}(A)}$ be as in (iii) above. Assume that some positive multiple $\nu\bar{D}$ contains a big divisor coming from $\bar{A}$.

Then $\exists$ irred. comp. $E \subset D \cap X_0(\widehat{\exp}_A f)$ such that $\widehat{\exp}_A f(\mathbf{C}) \cap E$ is Zariski dense in E; in particular, $|\widehat{\exp}_A f(\mathbf{C}) \cap D| = \infty$.

N.B. For $\exp_A f : \mathbf{C} \to A$, by Corvaja-N. ('12), answering a problem in Lang's monog. '66.

The proof of the 2nd Main Thm. 3.1 is rather long but we carry out the proof along the way as for $\exp_A f : \mathbf{C} \to A$ (N.-Winkelmann-Yamanoi) by making use of Key Lem 2.3.

The next theorem says that the distribution $\widehat{\exp}_A f^* D$ on $\mathbf{C}$ contains an ample information of $\widehat{A}$, D and f; we have the following unicity theorem of H. Cartan–P. Erdös–K. Yamanoi type (cf. Yamanoi Forum Math. 2004, Corvaja-N. Math. Ann. 2012)

Thm. 3.3 (Unicity)**.** Let A_j $(j = 1, 2)$ be two semi-abelian varieties and let D_j $(j = 1, 2)$ be effective reduced A_j-big divisors on $\widehat{A}_j$ with

$$\widehat{\mathrm{St}}(D_j) := \{x \in \widehat{A}_j : x + D_j = D_j\} = \{0\}.$$

Let $f_j : \mathbf{C} \to \mathrm{Lie}(A_j)$ be A_j-nondegenerate. Assume that

$$\mathrm{Supp}\,(\widehat{\exp}_{A_1} f_1)^* D_1 = \mathrm{Supp}\,(\widehat{\exp}_{A_2} f_2)^* D_2.$$

Then $\exists\, \alpha : A_1 \xrightarrow{\cong} A_2$ with $\hat{\alpha} : \widehat{A}_1 \to \widehat{A}_2$, such that

- $\hat{\alpha}^* D_2 = D_1$,
- $\widehat{\exp}_{A_2} f_2 = \hat{\alpha} \circ \widehat{\exp}_{A_1} f_1$, up to translations of $\widehat{A}_j$.

Remarks to some extensions:

(i) $\mathbf{C} \Rightarrow \Delta(r)^*$ (isolated essential singularity, Big Picard type).

(ii) $\mathbf{C} \Rightarrow$ affine alg. curve.

(iii) $\mathbf{C} \Rightarrow$ (parabolic) Riemann suface with involving a counting function of Euler numbers.

(iv) Hyperbolic case?

Hyperbolic Bloch–Ochiai by "O-minimal", Pila, Ulmo, Mok (2018 at Kanazawa), ...

Thank you for your attention!!

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