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Bundle-convexity of locally pseudoconvex domains

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Abstract

The method of L^2 estimates for the $\overline{\partial}$ operator will be applied in such a way that the finite-dimensionality of certain L^2 $\overline{\partial}$-cohomology groups works to prove that a Hermitian holomorphic line bundle L over a complex manifold X is bimeromorphically equivalent to an ample bundle over a bounded locally pseudoconvex domain Ω if the curvature form of L is positive on $\partial\Omega$ and $\partial\Omega$ is either a C^2 real hypersurface or the support of an effective divisor whose normal bundle is semipositive.

$$\underset{\text{hol. l. bdle}}{L} \longrightarrow X \ \text{cx mfd} \ \Supset \underset{\text{loc. } \psi\text{cvx}}{\Omega}$$

Q $\quad L|_{\partial\Omega} > 0 \overset{?}{\Longrightarrow} \Omega$ is L^μ cvx for $\mu \gg 1$.

A $\quad$ Yes, if $\partial\Omega \in C^2$ or

$\partial\Omega = |D|$ for some eff. div. D on X

s.t. $N_D := [D]|_{|D|} \geq 0$.

A History of $L \xrightarrow{\pi} X$

0. Abel, Jacobi $\Rightarrow$ Weierstrass, Riemann

1. Poincaré, Weyl $\Rightarrow$ Kodaira

$L > 0$ and $X \supset\!\supset X$

$$\Rightarrow \quad \begin{array}{ccc} X & \subset & \Gamma(X, L^{\mu})^{*}/\mathbb{C}\setminus\{0\} \\ \Psi & & \Psi \\ x & \longmapsto & \{s \mid s(x) = 0\} \end{array}$$

for $\mu \gg 1$. (ampleness)

2. Oka

$$X \in \pi_0(\mathcal{O}_{\mathbb{C}^n}) \Rightarrow X \text{ is hol. cvx}$$

i.e.

$$\exists \gamma \in X^{\mathbb{N}} \text{ s.t. } \gamma(\mathbb{N}) \not\Subset X$$

$$\exists f \in \mathcal{O}(X) := \Gamma(X, \underset{\underset{1_X}{\parallel}}{\mathbb{C} \times X}) \text{ s.t.}$$

$$f(\gamma(\mathbb{N})) \not\Subset \mathbb{C}.$$

3. Grauert

$$1_X|_{\partial\Omega} > 0 \Rightarrow \Omega \text{ is hol. cvx.}$$

Definitions

I. L convexity in Grauert's sense ('63)

$\forall K \Subset X \ \exists \hat{K} \Subset L$ s.t. $\forall x \in X \setminus \pi(\hat{K}) \ \forall v \in \pi^{-1}(x)$

$\exists s \in \Gamma(X, L)$ s.t. $s(K) \subset \hat{K}$ and $s(x) = v$.

II. Stein $\iff$ 1-complete

$\iff$ hol. cvx (1_X cvx) and

$$\begin{array}{ccc} X & \hookrightarrow & \Gamma(X, 1_X)^* / \mathbb{C} \setminus \{0\} \quad (1_X \text{ sep}) \\ \Psi & & \Psi \\ x & \longmapsto & \{f \mid f(x) = 0\} \end{array}$$

Definitions for $\Omega \underset{\text{loc. } \psi\text{cvx}}{\subset\subset} X$

III. L cvx $:\Leftrightarrow$ $\forall \gamma \in \Omega^{\mathbb{N}}$ s.t. $\gamma(\mathbb{N}) \not\subset\subset \Omega$
$\exists s \in \Gamma(\Omega, L)$ s.t. $s(\gamma(\mathbb{N})) \not\subset\subset L$.

IV. L cplt $:\Leftrightarrow$ L cvx and

$$\begin{array}{ccc} \Omega & \hookrightarrow & \Gamma(\Omega, L)^* / \mathbb{C} \setminus \{0\} \\ \Psi & & \Psi \\ x & \longmapsto & \{s \mid s(x) = 0\} \end{array}$$

(very ample)

Extensions of Oka '53, Kodaira '53 and Grauert '58

1. Kohn, Andreotti-Vesentini, Hörmander (Bergman)
2. Akizuki-Nakano, Grauert-Riemenschneider
3. Kawamata, Viehweg, Siu, Demailly, Nadel, Tsuji
4. O'79, Nakano-Rhai, Abdelkader, Takayama

'93 Pinney : $L > 0 \Rightarrow \Omega$ is $L^{\mu \gg 1}$-cvx if $\partial\Omega \in C^2$ and $X \Subset X$.

'95 Asserda : $\sim$ if $X \Subset X$.

Some activities related to the L^2 extension

L^2 estimates for $\overline{\partial}$ $\underset{Ann.of Math.'95}{\longrightarrow}$ Bergman metric

$\swarrow \qquad \searrow \qquad\qquad\qquad \searrow$

$\swarrow$ Math. Z. '87 $\searrow$ MRL '20 ($\xrightarrow{Abh.MSUH'21}$) $\xrightarrow{'22}$ bundle-convexity

L^2 extension $\underset{KJM'20}{\longrightarrow}$ Nishino's rigidity $\qquad \searrow$ PRIMS '22

$\approx \qquad\qquad\qquad\qquad H^{p,q}_{alg}(\Omega, L)$

$\log B_{\Omega_t}(z,z) \in PSH\ w.r.t.\ (t,z)$ $\qquad$ $\approx$ Zou, Watanabe, Rao

Maitani-Yamaguchi, Guan-Zhou, Błocki, Berndtsson-Lempert, Păun-Takayama.

Convexity notions and kernel asymptotics : an overview

Recall that a domain Ω over $\mathbb{C}^n$ is said to be a **domain of holomorphy** if Ω is equivalent to a connected component of the structure sheaf $\mathcal{O}(=\mathcal{O}_{\mathbb{C}^n}) \to \mathbb{C}^n$, i.e.

$$\Omega \in \pi_0(\mathcal{O}).$$

Every holomorphic map

$$\{(z,w) \in \mathbb{D}^2; |z| > \frac{1}{2} \text{ or } |w| < \frac{1}{2}\} \to \Omega \in \pi_0(\mathcal{O}) \quad (\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\})$$

is extendable to a holomorphic map $\mathbb{D}^2 \to \Omega$. (Hartogs 1906)

Every domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2-smooth boundary has a defining function ρ whose complex Hessian

$$\partial\overline{\partial}\rho = \sum_{1 \le j,k \le n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} dz_j d\overline{z}_k$$

is positive semi-definite on the complex tangent spaces of $\partial\Omega$. (Levi 1910)

$\partial\overline{\partial}\rho|_{\mathrm{Ker}\,\partial\rho}$ is called the **Levi form**. A domain Ω with C^2-smooth boundary is called **strongly pseudoconvex** if Ω has a defining function whose Levi form is positive definite on $\partial\Omega$.

Holomorphic convexity was introduced by Cartan and Thullen. A complex manifold X is called **holomorphically convex** if X can be mapped onto a closed complex analytic subset of $\mathbb{C}^N$ by a proper holomorphic map, or X satisfies

$$\forall \gamma \in X^{\mathbb{N}} \ s.t. \ \gamma(\mathbb{N}) \not\Subset X \ \exists f \in \mathcal{O}(X) \ s.t. \ f(\gamma(\mathbb{N})) \not\Subset \mathbb{C}.$$

Here $\mathcal{O}(X) := \{f \in C^1(X); \overline{\partial} f = 0\}$.

$$\Omega \underset{hol.cvx}{\in} \pi_0(\mathcal{O}_{\mathbb{C}^n}) \Rightarrow H^1(\Omega, \mathcal{O}) = 0. \quad (\text{Oka } '37)$$

A domain Ω over X is said to be **locally pseudoconvex** if every point of X has a neighborhood U such that the preimage of U in Ω is holomorphically convex.

Relations between these convexity notions have been clarified by Oka based on the study of **plurisubharmonic(=psh)** functions.

$$\Omega \in \pi_0(\mathcal{O}_{\mathbb{C}^n}) \Rightarrow \Omega \underset{hol.cvx}{\in} \pi_0(\mathcal{O}_{\mathbb{C}^n}) \quad (\text{Oka } '42, '43, '53)$$

Every $\Omega \in \pi_0(\mathcal{O}_{\mathbb{C}^n})$ admits a plurisubharmonic exhaustion function.

A complex manifold M equipped with a C^∞ psh exhaustion function is called **weakly 1-complete (=w.1.c)**. A w.1.c. manifold (M,φ) is called **1-convex** if $\partial\bar{\partial}\varphi > 0$ outside a compact subset of M.

$$\Omega \in \pi_0(\mathcal{O}_{\mathbb{C}^n}) \qquad \Rightarrow \qquad Spec(\mathcal{O}(\Omega)) \underset{closed}{\subset} \mathbb{C}^N \tag{1}$$

$\searrow$ Oka '42,'53 $\nearrow$ Grauert '58 (Nishino '61)

Ω is 1-convex.

The following suggests that the above diagram can be refined :

$$\Omega \underset{hol.cvx}{\Subset} \mathbb{C}^2 \text{ and } \partial\Omega \in C^2 \Rightarrow \delta(z)^{-2} \lesssim B_\Omega(z,z) \lesssim \delta(z)^{-3} \quad \text{(Bergman'33)}. \tag{2}$$

Here $B_\Omega(z,w)$ denotes the Bergman kernel function of Ω.

Geometry of $\partial\Omega$ $\longrightarrow$ Analysis of $\mathcal{O}(\Omega)$

In 1965, Hörmander proved that, given a domain $\Omega \subset \mathbb{C}^n$,

$$\lim_{z \to z_0} B_\Omega(z,z)\delta_\Omega(z)^{n+1} \textit{ exists and } > 0$$

if the range of the $\overline{\partial}$-operator $L^{0,0}_{(2)}(\Omega) \to L^{0,1}_{(2)}(\Omega)$ is closed and $\partial\Omega$ is strongly pseudoconvex at z_0. Here $L^{p,q}_{(2)}(\Omega)$ denotes the space of L^2 (p,q)-forms on Ω.

In 1974, Fefferman proved for strongly pseudoconvex domains Ω with C^∞-smooth boundary that

$$B_\Omega(z,z) = \varphi(z)\delta_\Omega(z)^{-n-1} + \psi(z)\log\delta(z)$$

holds for some C^∞ functions φ and ψ on $\overline{\Omega}$. The following is an application.

Fefferman's theorem. Every biholomorphic map between two strongly pseudoconvex bounded domains Ω_1 and Ω_2 with C^∞-smooth boundary extends as a diffeomorphism from $\overline{\Omega_1}$ to $\overline{\Omega_2}$.

By an L^2 extension theorem, (Ohsawa-Takegoshi 1987)

$$\Omega \underset{\psi cvx}{\Subset} \mathbb{C}^n \text{ and } \partial\Omega \in Lip \Rightarrow \delta_\Omega(z)^{-2} \lesssim B_\Omega(z,z).$$

Remark

$\partial\Omega \in$ Hölder $\Rightarrow$ $\lim_{z \to \partial\Omega} B_\Omega(z,z) = \infty$

(Chen B.-Y. '21)

L^2 estimates on complete manifolds

(M, g): a **complete** Hermitian manifold of dimension n.
$\omega = \omega_g$:=the fundamental form of g.
$\Lambda := \omega^* =$ the adjoint of $u \mapsto \omega \wedge u$.
K_M:=the canonical line bundle of M.
(E, h): a Hermitian holomorphic vector bundle over M.
$C^{p,q}(M, E)$:=$\{E$-valued C^∞ (p, q)-forms on $M\}$ $(= C^{0,q}(M, K_M \otimes E)$ if $p = n)$.
$\Theta = \Theta_h :=$ the curvature form of h.
$\Theta \in C^{1,1}(M, E^* \otimes E), \quad h \circ \Theta \in C^{0,0}(M, (E \otimes T_M)^* \otimes \overline{(E \otimes T_M)^*}))$.
$\Omega \underset{open}{\subset} M$, $C_0^{p,q}(\Omega, E) := \{u \in C^{p,q}(\Omega, E); \mathrm{supp}\, u \Subset \Omega\}$.

Basic Estimate I. $d\omega|_\Omega = 0 \;\Rightarrow\; (\sqrt{-1}\Theta\Lambda u, u) \le \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$ for all $u \in C_0^{n,q}(\Omega, E)$ $(q \ge 1)$.

$L^{p,q}_{(2)}(\Omega, E) = L^{p,q}_{(2)}(\Omega, E, g, h) :=$ the completion of $C^{p,q}_0(\Omega, E)$ w.r.t. $\|\cdot\|$.
$L^{p,q}_{(2),\varphi}(M, E) := L^{p,q}_{(2)}(M, E, g, e^{-\varphi}h)$, $\|u\|^2_\Omega := \int_\Omega |u|^2 dV \quad (dV := \frac{1}{n!}\omega^n)$
$L^{p,q}_{(2),loc}(\Omega, E) :=$ {locally square integrable forms}.

$$H^{p,q}_{(2)}(M, E) = H^{p,q}_{(2)}(M, E, g, h) := \frac{\mathrm{Ker}\overline{\partial} \cap L^{p,q}_{(2)}(M, E)}{\overline{\partial}(L^{p,q-1}_{(2)}(M, E)) \cap L^{p,q}_{(2)}(M, E)}.$$

$$H^{p,q}(M, E) := \frac{\mathrm{Ker}\overline{\partial} \cap C^{p,q}(M, E)}{\overline{\partial}(C^{p,q-1}(M, E))} \cong \frac{\mathrm{Ker}\overline{\partial} \cap L^{p,q}_{(2),loc}(M, E)}{\overline{\partial}(L^{p,q-1}_{(2),loc}(M, E)) \cap L^{p,q}_{(2),loc}(M, E)}.$$

Theorem 1. (Kodaira-Nakano-Andreotti-Vesentini) *$H^{n,q}_{(2)}(M, E) = 0$ for $q \geq 1$ if $\sqrt{-1}\Theta > cId_E \otimes \omega$ for some $c > 0$ and $d\omega|_{M\setminus\Omega} = 0$ for some $\Omega \Subset M$.*

Theorem 2. (Hörmander-Demailly)

$\Omega \Subset M$, $d\omega|_{M\setminus\Omega} = 0$ *and* $\Theta > 0$ $(\iff: \ E > 0) \ \Rightarrow$
$\forall v \in L^{n,1}_{(2)}(M, E)$ *s.t.* $\overline{\partial}v = 0$ *and* $((\sqrt{-1}\Theta\Lambda)^{-1}v, v) < \infty$,
$\exists u \in L^{n,0}_{(2)}(M, E)$ *s.t.* $\overline{\partial}u = v$ *and* $\|u\|^2 \leq ((\sqrt{-1}\Theta\Lambda)^{-1}v, v)$.

See also PRIMS '80 and PRIMS'84.

Theorem 3. (Akizuki-Nakano-Andreotti-Vesentini)

$\Theta = g$ (rank$E = 1$) $\Rightarrow$ $H^{p,q}_{(2)}(M, E, g, h) = 0$ *for* $p + q > n$.

Theorem 4. (Ohsawa '22) $X \Subset X$, $D \underset{\textit{smooth div.}}{\subset} X$, rank$E = 1$, $E > 0$ *and* $[D]|_D \geq 0$

$\Rightarrow$ $H^{p,q}_{alg}(X \setminus D, E) = 0$ *for* $p + q > n$.

Basic Estimate II. $\Omega \Subset M$ and $d\omega|_{M\setminus\Omega} = 0$ $\Rightarrow$

$\exists C$ s.t. $(\sqrt{-1}\Theta\Lambda u, u) \leq C(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 + \|u\|^2_\Omega)$ for all $u \in C^{n,q}_0(M, E)$ $(q \geq 1)$.

Theorem 5. (Hörmander)

$$\dim H^{n,q}_{(2)}(M, E) < \infty \textit{ for } q \geq 1$$

if $\exists \Omega \Subset M$ *s.t.*
$d\omega|_{M\setminus\Omega} = 0$ *and* $\sqrt{-1}\Theta - cId_E \otimes \omega > 0$ *on* $M \setminus \Omega > 0$ *for some* $c > 0$.

Generalizations of Oka theory by L^2 method

(M, φ) : w.1.c. $M_c := \varphi^{-1}((-\infty, c)) \Subset M$.

Theorem 6. (Nakano-Kazama) *Given* (M, φ, E, h) *with* $\Theta > 0$,

$$H^{n,0}(M_c, E) = \overline{H^{n,0}(M, E)|_{M_c}}$$

holds for any c *and* $H^{n,q}(M, E) = H^{n,q}(M_c, E) = 0$ *holds for any* c *and* $q \geq 1$.

Theorem 7. (Ohsawa-Nakano-Rhai-Abdelkader) *If* (M, φ, g, E, h, c) *satisfies* $d\omega|_{M \setminus M_c} = 0$ *and* $\Theta|_{M \setminus M_c} > 0$, *then*

$$H^{n,0}(M_c, E) = \overline{H^{n,0}(M, E)|_{M_c}},$$

$$\dim H^{n,q}(M, E) < \infty$$

and

$$H^{n,q}(M, E) \cong H^{n,q}(M_c, E) \ for\ q \geq 1.$$

Equivalence of the following 1) and 2) is a consequence of Theorem 7.

1) $\forall d > c \; \exists \; m \in \mathbb{N}$ and a meromorphic map η from M_d to $\mathbb{CP}^N$ ($N \gg 1$) by the ratio of holomorphic sections of $L^m|_{M_d}$ s.t. $\eta|_{M_d \setminus M_c}$ is a holomorphic embedding.

2) $\Theta|_{M \setminus M_c}$ is positive.

Weakly 1-complete manifolds need not be holomorphically convex. Nevertheless, Theorem 7 implies as above an existence theorem analogous to Grauert'58 for sections of vector bundles on weakly 1-complete manifolds. Based on such a result, Grauert'58 can be extended as follows.

Theorem 8. *If a w.1.c. manifold* (M, φ) *satisfies* $K_M|_{M \setminus M_c} < 0$ *for some* c, M *is holomorphically convex.*

Takayama '98 proved Theorem 8 under the assumption $K_M < 0$. The general case is obtained similarly in view of the equivalence of the above 1) and 2).

Analysis $H^{n,1}_{(2)} = 0 \Rightarrow H^{n,0}_{(2)} \neq 0$

$$x \in X \Supset \underset{\text{nbd}}{U} \ni x$$

$$\psi : X \xrightarrow[C^0]{} [-\infty, -1]$$

$$\int_U e^{-\psi} = \infty \quad \text{and} \quad x = \psi^{-1}(-\infty)$$

$$\Rightarrow \quad \dim H^{n,1}_{(2)}(X, g, e^{-\psi}) + \dim H^{n,0}_{(2)}(X) > 0$$

$$\dim H^{n,1}_{(2)} < \infty \Longrightarrow H^{n,0}_{(2)} \neq 0$$

$X \supset \Gamma = \overline{\Gamma}$, $\#\Gamma = \infty$

$\exists\, U \underset{\text{nbd}}{\supset} \Gamma$ s.t. $\pi_0(U) = \{U_1, U_2, \cdots\}$,

$\#(U_k \cap \Gamma) = 1$ and $\exists\, \psi : X \xrightarrow[C^0]{} [-\infty, -1]$

s.t. $\displaystyle\int_{U_k} e^{-\psi} = \infty$ and $U \setminus \Gamma = \psi^{-1}((-\infty, -1))$

$$\Longrightarrow \dim H^{n,1}_{(2)}(X, g, e^{-\psi}) + \dim H^{n,0}_{(2)}(X) = \infty$$

Geometry

$$\Omega \underset{str.\psi cvx}{\Subset} M \approx \Omega \underset{loc.\psi cvx}{\Subset} M \text{ and } E|_{\partial\Omega} > 0$$

$$M \Subset M \;----\; w.1.c. \;----\; M = Spec\mathcal{O}(M)$$

$$K_M > 0 \cdots K_M = 0 \cdots K_M < 0$$

$$\Omega \Subset \Omega \;--\; \Omega \underset{loc.\psi cvx}{\Subset} M \; and \; E|_{\partial\Omega} > 0 \;--\; \partial\Omega \in SPC$$

(geometry of $\Omega \Subset M$ from the viewpoint of analytic continuation)

Results on locally pseudoconvex domains

$\Omega \underset{str.\psi cvx}{\Subset} M$ is an intrinsic property of Ω (Nakano-Ohsawa), but $\Omega \underset{loc.\psi cvx}{\Subset} M$ is not.

Examples.

1. $\mathbb{CP}^2 \underset{not\ loc.\psi cvx}{\Supset} \Omega = \mathbb{CP}^2 \setminus \{p\} \cong \mathcal{O}(1)_{\mathbb{CP}^1} \underset{loc.\psi cvx}{\subset} \mathbb{CP}^2$ blown-up at p.

2. $(\mathbb{C}^n \setminus \{0\}) \times \{\zeta \in \mathbb{C}; 1 < |\zeta| < \exp(2\pi^2/\log 2)\}$

$\cong \Omega \underset{loc.\psi cvx\ and\ \partial\Omega \in C^\omega}{\subset} (\mathbb{C}^n \setminus \{0\}) \times \mathbb{CP}^1/\langle (z,\zeta) \mapsto (2z, 2\zeta)\rangle.$

Ohsawa '82 ($n = 1$) and Diederich-Fornaess '82 ($n \geq 2$).

Theorem 9. (Diederich-Ohsawa '85) $M \underset{Kaehler}{\Subset} M$ *and* $\rho \in Hom(\pi_1(M), \mathrm{Aut}\mathbb{D})$

$\Rightarrow\ \Omega = M \times_\rho \mathbb{D} \underset{w.1.c.}{\Subset} M \times_\rho \mathbb{CP}^1.$

Bundle-convexity theorems

Definition 1. (Grauert '63) *Given $E \to M$, M is called* ***strictly E-convex*** *if*

$$\forall K \Subset M\ \exists \hat{K} \Subset E\ s.t.\ \forall x \in M \setminus 0^{-1}(\hat{K})\ and\ \forall v \in E_x$$

$\exists s \in \Gamma(M, E)$ *s.t.* $s(K) \subset \hat{K}$ *and* $s(x) = v$. *(0 : $M \to E$ denotes the zero section.)*

Definition 2. (Pinney '92, Asserda '95) *Given $E \to M$ and $\Omega \Subset M$, Ω is called* ***E-convex*** *if*

$$\forall \gamma \in \Omega^{\mathbb{N}}\ s.t.\ \gamma(\mathbb{N}) \not\Subset \Omega\ \exists s \in \Gamma(\Omega, E)\ s.t.\ |s(\gamma(\mathbb{N}))| \not\Subset \mathbb{R}.$$

Theorem 10. (Asserda '95) $M \Subset M$, $\operatorname{rank} E = 1$, $E > 0$ *and* $\Omega \underset{loc.\psi cvx}{\Subset} M$

$\Rightarrow \Omega$ *is E^{μ}-convex for $\mu \gg 1$.*

Theorem 11. **(bundle-convexity I)** $\Omega \underset{loc.\psi cvx}{\Subset} M$, $\partial\Omega \in C^2$, $E \to M$, $\mathrm{rank} E = 1$ *and* $E|_{\partial\Omega} > 0 \quad \Rightarrow \quad \Omega$ *is* E^μ*-convex for* $\mu \gg 1$.

Theorem 12. **(bundle-convexity II)** $\Omega \underset{loc.\psi cvx}{\Subset} M$, $\partial\Omega = |D|$ *for some effective divisor* D *on* M *s.t.* $[D]|_{|D|} \geq 0$, $E \to M$, $\mathrm{rank} E = 1$ *and* $E|_{\partial\Omega} > 0$
$\Rightarrow \quad \Omega$ *is* E^μ*-convex for* $\mu \gg 1$.

Theorem 13. (Ohsawa '22) *In the situation of Theorem 11 or Theorem 12, assume that* $E = K_M^{-1}$. *Then* Ω *can be mapped holomorphically and properly onto a locally closed analytic set in* $\mathbb{C}^N$.

It is likely that Ω is holomorphically convex.

A result on the kernel asymptotics for the case $\partial\Omega \in C^2$ is the following.

Theorem 14. *Let* Ω *be a bounded locally pseudoconvex domain with* C^2*-smooth boundary in a complex manifold* M *and let* $E \to M$ *be a holomorphic line bundle with a* C^∞ *fiber metric* h *whose curvature form is positive at every point of* $\partial\Omega$. *Then, for any* $\varepsilon > 0$ *one can find* $\nu_0 \in \mathbb{N}$ *such that*

$$\liminf_{z \to \partial\Omega} B_{\Omega, E^\nu}(z) \cdot \rho(z)^{2-\varepsilon} > 0$$

holds for any $\nu \geq \nu_0$. *Here* B_{Ω,E^ν} *denotes the Bergman kernel for the* L^2 E^ν*-valued holomorphic* n*-forms with respect to* h^ν.

Proof of bundle-convexity I. Finite-dimensionality of the L^2 $\bar{\partial}$-cohomology with respect to a complete metric on $\Omega \setminus \gamma(\mathbb{N})$ for a class of $\gamma : \mathbb{N} \to \Omega$ is applied. More precisely, for any $z_0 \in \partial\Omega$ one can find a sequence $\gamma \in \Omega^{\mathbb{N}}$ with $\lim_{k\to\infty} \gamma(k) = z_0$, a complete metric g on $\Omega \setminus \gamma(\mathbb{N})$, $\psi : \Omega \to [-\infty, -1]$ with $\psi^{-1}(-\infty) = \gamma(\mathbb{N})$ and $-\partial\bar{\partial} \log(-\psi) \approx g$ near $\gamma(\mathbb{N})$ such that

$$\dim H^{n,1}_{(2)}(\Omega \setminus \gamma(\mathbb{N}), E^{\mu^2}, g, h^{\mu^2} e^{-\psi}(-\psi)\delta^{\mu}_{\Omega}) < \infty. \tag{3}$$

One can apply (3) to find desired sections by choosing ψ so that $e^{-\psi}$ is non-integrable around any point of $\gamma(\mathbb{N})$.

Proof of bundle-convexity II. $\exists m \in \mathbb{N}$ s.t. $H^{0,q}(M, K_M \otimes E^m \otimes [D]^{\mu}) \to H^{0.q}(D, K_M \otimes E^m \otimes [D]^{\mu}|_D)$ for $\mu \gg 1$.

Proof of Theorem 14. After reducing the question to the case where $\Theta_h > 0$ on Ω allowing some singularity of h along $\partial\Omega$, such asymptotics can be analyzed by solving the $\bar{\partial}$ equations with L^2 norm estimates by a standard technique.

$$n = \dim X \Rightarrow H^{n,0}_{(2)}(X, L, g_1, h) = H^{n,0}_{(2)}(X, L, g_2, h)$$

$$\Gamma \subset X \quad \exists U \underset{\text{nbd}}{\supset} \Gamma \quad \text{s.t.} \quad \pi_0(U) = \{U_1, U_2, \cdots\}$$

$$\#(U_k \cap \Gamma) = 1, \quad U_k \subset\subset X, \quad \#\Gamma = \infty, \quad \overline{\Gamma} = \Gamma$$

Suppose that

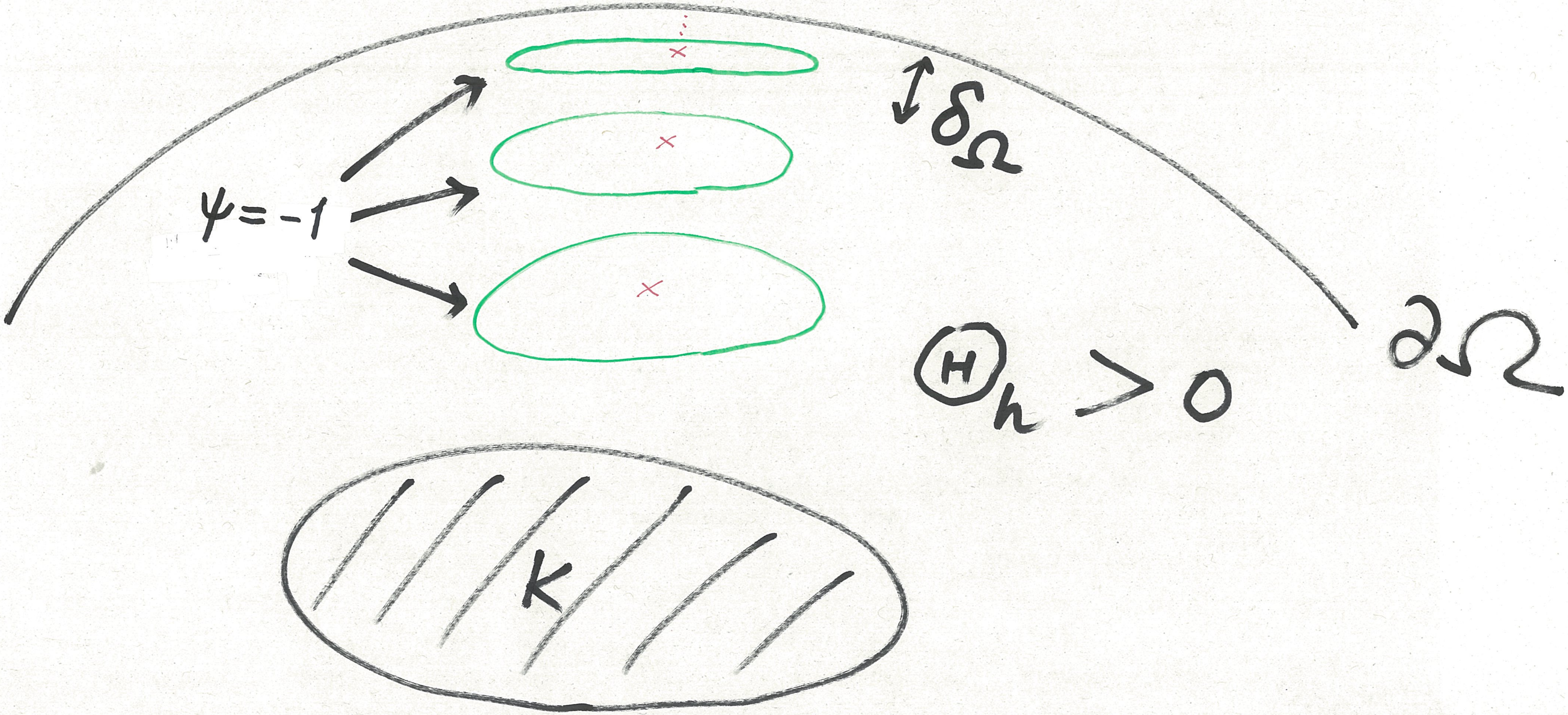
$\exists$ a cplt Herm. metric g on $X \setminus \Gamma$

$$\exists \psi : X \xrightarrow[C^0]{} [-\infty, -1) \quad \text{s.t.} \quad \int_{U_k} e^{-\psi} = \infty \quad k = 1, 2, \cdots$$

and $\sup_{X \setminus \Gamma} |\partial \psi|_g < \infty$, then

$$\dim H^{n,1}_{(2)}(X \setminus \Gamma, L, g, he^{-\psi}) + \dim H^{n,0}_{(2)}(X, L, h) = \infty.$$

$$h^{\mu^2} e^{-\psi}(-\psi)\,\delta_\Omega^{\mu}$$



In Oka's theory, important existence theorems are tied together by an approximation theorem of Runge type, so that the existence of a psh exhaustion function is crucial to apply a limiting argument after approximating a given domain by strongly pseudoconvex subdomains. Hence

Oka, Grauert$\rightarrow$ Bergman, Hörmander$\rightarrow$ Fefferman, et al.

$\approx$

restricted class of psh exhaustions $\Rightarrow$sharper analytic results

If one wants to study the Bergman kernels on locally pseudoconvex domains in complex manifolds, a natural method is to apply the L^2 method **by generalizing it to the situation where the domain does not admit psh exhaustion functions in canonical ways**.

Open questions. Given a Kähler manifold M,

Q1. $\Omega \underset{loc.\psi cvx}{\Subset} M \underset{?}{\Rightarrow} \Omega$ is w.1.c.

Q2. $M \Subset M$ and $\tilde{M} \underset{covering}{\rightarrow} M \underset{?}{\Rightarrow} \tilde{M}$ is w.1.c.

Hence, Oka's approach has to be replaced by a more direct one.

Notes on Theorem 13

1. $\exists\ X \longrightarrow \mathbb{C}^2$ s.t.

   loc. Stein but not Stein

   $\left(X \overset{?}{\subset} \mathbb{C}^N \right)$ (Fornaess '77)

   loc. closed

2. $\exists\ X \longrightarrow \mathbb{C}^3$ s.t.

   loc. Stein but not Stein

   $X \subset \mathbb{C}^N$ (Colţoiu-Diederich '07)

   loc. closed

Thanks a lot !

以上です