How to Find Subvarieties Obstracting Kobayashi Hyperbolicity – An Elementary Method and its Variation –

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Sequence of Kodaira Embeddings and Grassmannians

• Let X be an n-dimensional projective manifold and $L \to X$ an ample line bundle. For $\forall m \in \mathbb{Z}$ sufficiently large mL is very ample.

• The Kodaira map $\Phi_m : X \to |mL|^*$ is defined by

$$\Phi_m(x) = \{[s] \in |mL| \,|\, s(x) = 0\} \in |mL|^* \ .$$

For $\forall \mu \in \mathbb{G}(n, |mL|)$, the projection $\Phi_m^\mu : X \to \mu^* \cong \mathbb{P}^n$ is canonically defined. A $\mu \in \mathbb{G}(n, |mL|)$ determines the "linear center" $Z^{N_m - n - 1} \subset \mathbb{P}^{N_m} = |mL|^*$ and the projection is realized as a projection from $Z^{N_m - n - 1}$ to any $\mathbb{P}^n \subset \mathbb{P}^{N_m}$ disjoint from $Z^{N_m - n - 1}$.

• Thus we have the sequence of Kodaira maps $\{\Phi_m : X \to |mL|^*\}_m$ and Grassmannians $\{\mathbb{G}(n, |mL|)\}_m$. For each m, we have the collection of projections $\{\{\Phi_m^\mu : X \to \mu^* \cong \mathbb{P}^n\}_{\mu \in \mathbb{G}(n, |mL|)}\}_m$.

• Question : What happens if we compare theories on $\mathbb{P}^n$ with those on X via the sequence $\{\{\Phi_m^\mu : X \to \mu^* \cong \mathbb{P}^n\}_{\mu \in \mathbb{G}(n, |mL|)}\}_m$?

1-st Variation : Peak Section

• Set Up : Let (L, h) be an ample line bundle with a positive Hermitian metric, i.e., $c_1(L, h)$ is a Kähler form on X.

• Working on $\mathbb{P}^n$: We define a sequence of the Kodaira maps $\Phi_m : X \to |mL|$ where $H^0(X, mL)$ is equipped with the Fubini-Study metric determined by the L^2-inner product on $H^0(X, mL)$. We pick $x \in X$ and take an element s_x^μ of $\mathcal{O}_{\mathbb{P}^n}(1)$ with L^2-norm 1 which takes its peak at the image $\Phi_m^\mu(x) \in \mathbb{P}^n$. We can do this in the following way. Choose a unitary affine coordinate system centered at $\Phi_m^\mu(x)$. Then we have a polar set $P \cong \mathbb{P}^{n-1}$ of the point $\Phi_m^\mu(x)$. Then we take an element $s \in \mathcal{O}_{\mathbb{P}^n}(1)$ s.t. $(s) = P$. Then $|s|_{\mathrm{FS}}$ has its peak at $\Phi_m^\mu(x)$ and therefore $s = s_x^\mu$.

1-st Variation : Peak Section (continued)

• Comparison with X : For the purpose of comparison, we vary $\mu \in \mathbb{G}(n, |mL|)$ and sum up $(\Phi_m^\mu)^* s_{x,\mu}$ to define

$$\sigma_x := \int_{\mu \in \mathbb{G}(n,|mL|)} (\Phi_m^\mu)^* s_{x,\mu} dV_{\mathrm{Haar}} \ .$$

Here we have to multiply appropriate $e^{i\theta}$ before integration in order to avoid cancellation. We thus have a section $\sigma_x \in H^0(X, mL)$ which has a peak at a given point $x \in X$.

• Thus the comparison method explains the Tian-Zelditch Peak Section Theorem.

1-st Variation : Peak Section (continued)

• Peak Section on $\mathbb{P}^n$: In the case of $\mathcal{O}_{\mathbb{P}^n}(m)$ (m large), we can explicitly construct a peak section by expressing a given point of $\mathbb{P}^n$ as an intersection of various $(S^1)^n$ orbit obtained by varying the embedding of $(S^1)^n$ into the maximal compact subgroup $\mathrm{SU}(n+1) \subset \mathrm{SL}(n+1)$ determined by the Fubini-Study structure.

• Concentration : The large m in $\mathcal{O}_{\mathbb{P}^n}(m)$ corresponds to the large m in $|mL|$ so that the projection $\Phi_m^\mu : X \to \mathbb{P}^n$ has high degree L^m and therefore an element of $\mathcal{O}_{\mathbb{P}^n}(1)$ is pulled-back via Φ_m^μ to a section of mL and therefore the concentration at x becomes stronger.

2-nd Variation : Shiffman-Zelditch Approximation

Theorem (Shiffman-Zelditch Approximation Theorem)

Let $(L, e^{-\varphi})$ be a positive line bundle s.t. $\omega = dd^c\varphi$ is a Kähler form. Let $[s_m] \in |mL|$ be a Haar distributed random section. Then the integration current $\frac{1}{m}(s_m)$ almost surely converges to the Kähler form ω as $m \to \infty$.

- We consider the following situation : Let $R_\mu \subset X$ be the ramification divisor of the projection $\mu : X \to \mathbb{P}^n$. Then the Riemann-Hurwitz Formula implies

$$[R_\mu] = \mu^* K_{\mathbb{P}^n}^{-1} - K_X^{-1}$$

and the line bundle $F_m = [R_\mu]$ is independent of $\mu \in \mathbb{G}(n, |mL|)$.

- Let $(F_m, e^{-\varphi_m})$ be a sequence of positive line bundles s.t. $\omega_m := \frac{1}{(\deg F_m)^{1/n}} dd^c\varphi_m$ converges to a Kähler form ω_∞.

2-nd Variation : Shiffman-Zelditch Approximation (continued)

In the set up of the previous page, we have the following variant of the Shiffman-Zelditch Approximation Theorem (which is based on the Tian/Zelditch Peak Section Theorem).

Observation (M. Izumi)

The ramification divisor R_μ for Haar distributed random projection $\mu \in \mathbb{G}(n, |mL|)$ almost surely converges to the uniform distribution determined by the Kähler form ω_∞ as $m \to \infty$.

$$\begin{array}{ccc}
\mathbb{G}(n, |mD|) & \xrightarrow{\text{Plücker embedding}} & \mathbb{P}(\bigwedge^{n+1} H^0(X, \mathcal{O}_X(mD))) \\
= \big\downarrow & & \big\downarrow \text{zero set on } X \\
\mathbb{G}(n, |mD|) & \xrightarrow{\text{ramification divisor}} & \mathbb{P}(H^0(X, \mathcal{O}_X(R_\mu)))
\end{array}$$

The bottom arrow lifts to $\mathrm{Aut}(\mathbb{G}(n, |mL|)) \to \mathrm{Aut}(\mathbb{P}(H^0(X, \mathcal{O}_X(R_\mu)))$.

3-rd Variation : Geometric Quantization

• Let X be a projective manifold and $(L, e^{-\varphi})$ a positive line bundle on X, i.e., $\omega := dd^c\varphi$ is a Kähler form representing $c_1(L)$. The concept of the real polarization was proposed to identify the Hilbert space $H^0(X, mL)$ (m being large) in the manner independent of the complex structure which makes ω Kähler. The 3-rd variation is concerned with this concept.

• The space $H^0(X, \mathcal{O}(mL))$ has L^2-inner product. Its group of symmetry is the unitary group. The Hermitian structure of $H^0(X, \mathcal{O}(mL))$ canonically induces the Fubini-Study structure on $|mL|$ which becomes homogeneous w.r.to the action of the unitary group. The Kodaira map embeds X into the Fubini-Study space $|mL|^* = \mathbb{P}^{N_m}$. A choice of a unitary basis determines the moment polytope. The unitary basis consists of elements of $\mathcal{O}_{\mathbb{P}^{N_m}}(1)$ and a vertex of the moment polytope corresponds to the points where the Fubini-Study norm of the corresponding basis element takes its maximum. The vertices consists of $(N_m + 1)$ points in $|mL|^* = \mathbb{P}^{N_m}$.

3-rd Variation : Geometric Quantization (continued)

• A choice μ of $(n+1)$ points from (N_m+1) vertices determines a $\mathbb{P}^n$ in $\mathbb{P}^{N_m}$. This is a discrete analogue of the Grassmannian $\mathbb{G}(n, |mL|)$. Interpreting this $\mathbb{P}^n$ as an element of $\mathbb{G}(n, |mL|)$, we consider the projection $\mu : X \to \mathbb{P}^n$. As soon as we consider μ, we get $(n+1)$ clusters of points each consisting of $\deg(\mu)$ points on X. Each cluster of points on X obtained this way corresponds to one of (N_m+1) vertices of the large moment polytope. Therefore, considering all such μ's, we get (N_m+1) clusters of $\deg(\mu)$ points on X.

3-rd Variation : Geometric Quantization (continued)

- The effect of large m : Suppose that m is very large. Then each of the special $(n+1)$ points of $\mathbb{P}^n$ corresponds to the maximum norm point of the corresponding unitary basis element of $\mathcal{O}_{\mathbb{P}^{N_m}}(1)$. Therefore, the pull-back via $\mu : X \to \mathbb{P}^n$ (note that $\deg(\mu)$ is large) is a section (say, $\sigma_{\mu,1}$) of mL which has peak along the $\deg(\mu)$ points in the corresponding cluster.

- Question 1. The magnitude of the cluster indefinitely becomes large. Can we trace the evolution of m-dependent "particular cluster" as $m \to \infty$? If so, does the particular cluster GH converges to a Lagrangian subspace of the Kähler manifold (X, ω) ?

- Question 2. Suppose that Question 1 is affirmative. Then we ask : Is the sequence of holomorphic sections (say, $\{\sigma_{\mu,1}\}_m$) asymptotically covariant constant (after scaling) along the Lagrangian subspace ?

3-rd Variation : Geometric Quantization (continued)

• Working on $\mathbb{P}^n$: Pick an m_0 s.t. m_0L is very ample and consider the sequence of m consisting of numbers divisible by m_0. In this setting we consider the previously defined collection of $\mathbb{P}^n$'s in $\mathbb{P}^{N_{m_0}}$. We consider the image of X and any one of $\mathbb{P}^n$ (considered in the above procedure for $m = m_0$) in the same ambient space $|mL|^* = \mathbb{P}_{\mathbb{N}_m}$. Then, on the image of $\mathbb{P}^n$ (taken originally from $|m_0L|^*$) in $|mL|^*$ (so the image has large degree in $|mL|^*$), there arises clusters via any chosen $\mathbb{P}^n$'s spanned by $(n+1)$ points among (N_m+1) vertices in $\mathbb{P}^{N_m}$.

3-rd Variation : Geometric Quantization (continued)

• Comparison between X and $\mathbb{P}^n$: We compare the clusters on $\mathbb{P}^n$ and those in X. The cluster in $\mathbb{P}^n$ approximates an $(S^1)^n$-orbit (a fiber of a moment map) and therefore asymptotically Lagrangian in GH sense. The metrized Kodaira maps are asymptotically isometric by the Tian/Zelditch Theorem. This implies that the sequence of clusters in X is also asymptotically Lagrangian. Taking various m_0 and restricting to the sequence of m divisible by m_0, we can trace the evolution of a particular m-dependent cluster and we get a sequence of sections of mL which is (after scaling) asymptotically covariant constant along the limit Lagrangian subspace.

• Although a Lagrangian fibration asymptoticaly appears on X, we can NOT say that, for a particular m (even very large), there exists a basis of $H^0(X, mL)$ each of which localizes covariant constantly along a limit Lagrangian subspace.

4-th Variation : Obstruction to Hyperbolicity

The 4-th variation is concerned with the attempt of characterizing a subvariety obstructing Kobayashi hyperbolicity

Theorem (Brody's criterion)

A compact complex manifold is not hyperbolic if and only if there exists a "complex line" $f : \mathbb{C} \to X$.

Therefore to find subvarieties obstructing Kobayashi hyperbolicity, we study a holomorphic curve $f : \mathbb{C} \to X$.

• Set Up : Let (X, D) be a pair of an n-dimensional projective manifold and a very ample s.n.c. divisor D. We set $L = [D]$. Let

$$f : \mathbb{C} \to X$$

be a holomorphic curve. Instead of the complete linear system $|mL|$, we consider a linear subsystem $|mL|'$ consisting of those whose divisor contains D.

4-th Variation : Obstruction to Hyperbolicity (continued)

1. Pick any n-dimensional linear subsystem $\mu \subset |mL|'$ and define $\Phi_m^\mu : X \to \mu^*$. Then, the map $\Phi_m^\mu : X \to \mu^*$ is realized as a projection $\mu : X \to \mathbb{P}^n$ (from the center Z^{N_m-n-1} determined by $|mL|$). We define $f_m^\mu := \Phi_m^\mu \circ f : \mathbb{C} \to \mathbb{P}^n$.

2. Let R_μ be the ramification divisor associated to $\Phi_m^\mu : X \to \mathbb{P}^n$. For a fixed m, the line bundle $[R_\mu]$ does not depend on μ and we denoted it by F_m. Then $\mathbb{G}' := \mathbb{G}(n, |mL|') \ni \mu \mapsto R_\mu \in |F_m|$ becomes a Haar distributed random variable.

3. As f_m^μ is $\mathbb{P}^n$-valued, we can use the affine coordinate system of $\mathbb{P}^n$ to define the Wronskian $W(j_n(f_m^\mu))$ as a $K_{\mathbb{P}^n}^{-1}$-valued holomorphic map. Here $j_n(f_m^\mu)$ denotes the n-th jet of f_m^μ in terms of the affine coordinates of $\mathbb{P}^n$ (we imagine the parallelepiped generated by $f', \dots, f^{(n)}$, where $f = (f_1, \dots, f_n)$).

4-th Variation : Obstruction to Hyperbolicity (continued)

4. Let $\psi_\mu \in H^0(X, F_m)$ be the defining section of R_μ, i.e. $R_\mu = (\psi_\mu)$.

5.

$$\Psi_\mu(j_n f) := \frac{W(j_n(f_m^\mu))}{\psi_\mu(f)} : \mathbb{C} \to \mu^* K_{\mathbb{P}^n}^{-1} \otimes F_m^{-1} = K_X^{-1}$$

is a K_X^{-1}-valued meromorphic map. Here we have used the Riemann-Hurwitz Theorem $\mu^* K_{\mathbb{P}^n}^{-1} = K_X^{-1} + R_\mu$.

6. From mD back to D : We consider $D + D'$ (s.n.c.) where $D' \in |(m-1)D|$ and take the mean over $|(m-1)D|$ of the theory, e.g., we take $\mathfrak{M}_{D' \in |(m-1)D|} m_{f, D+D'}(r)$.

4-th Variation : Obstruction to Hyperbolicity (continued)

• Question : Does the K_X^{-1}-valued meromorphic map $\frac{W(j_n(f_m^\mu))}{\psi_\mu(f)}$ play the same role as the Wronskian of a holomorphic curve in $\mathbb{P}^n$ (this is $K_{\mathbb{P}^n}^{-1}$-valued holomorphic map) ?

• Answer $(*)$: Yes, it does, modulo error term of magnitude $\varepsilon T_{f,E}(r)$. Here, $E \to X$ is any fixed ample line bundle on X and $\varepsilon > 0$ is any given positive number.

(Reason) Ahlfors-Yamanoi LLD + Measure Concentration.

Typical measure concentration phenomenon : Let k be fixed and d very large. Then, randomly chosen k vectors in $\mathbb{R}^d$ are almost surely orthogonal.

Digression on Nevanlinna Theory

X : a projective manifold.
$D = (\sigma)$: a s.n.c. divisor. $\{h = e^{-\varphi}\}$: a Hermitian metric of $[D]$.
$f : \mathbb{C} \to X$ is a non-constant holomorphic curve s.t. $f(0) \notin D$.

(proximity function) $m_{f,D}(r) = \int_0^{2\pi} \log \frac{1}{|\sigma(f(e^{i\theta}))|_h} \frac{d\theta}{2\pi}$.

(counting function) $N_{f,D}(r) = \int_0^r \frac{dt}{t} n_{f,D}(t)$,
$n_{f,D}(t) = \sharp\{z \in \mathbb{D}(t) \,|\, \sigma(f(z)) = 0\}$.

(order function) $T_{f,c_1([D])}(r) = \int_0^r \frac{dt}{t} \int_{\mathbb{D}(t)} dd^c\varphi$. homomorphism modulo $O(1)$.

(linear case : $X = \mathbb{P}^n$) $N_{f,\mathrm{Ram}}(r) = N_{W(f),S_0}(r)$.
Note : $W(f)$ is $K_{\mathbb{P}^n}^{-1}$-valued holomorphic.

(Poincaré-Lelong + Stokes $\Rightarrow$)
(First Main Theorem) $T_{f,c_1([D])}(r) = N_{f,D}(r) + m_{f,D}(r) - m_{f,D}(0)$.

4-th Variation : Obstruction to Hyperbolicity (continued)

• Comparison : The Cartan/Ahlfors Theory applied to $f_m^\mu : \mathbb{C} \to \mathbb{P}^n$:
(CA) $\quad m_{f_m^\mu, \mu(F)}(r) + N_{W(f_m^\mu), S_0}(r) \leq T_{f_m^\mu, K_{\mathbb{P}^n}^{-1}}(r) + S_f(r)$ $(F \in |mD|)$.

Plugging Rieman-Hurwitz $\mu^* K_{\mathbb{P}^n}^{-1} = K_X^{-1} + R_\mu$ to (CA), we get
$(X) \quad m_{f,D}(r) + \int_{\mu \in \mathbb{G}'} d\mu \{ N_{W(f_m^\mu), S_0}(r) - T_{f, R_\mu}(r) \} \leq T_{f, K_X^{-1}}(r) + S_f(r)$.
Here $S_f(r) = O(\log T_f(r) + \log r)$.

• Measure Concentration : Let $E \to X$ be any fixed ample line bundle.
We have the **key inequality** :
$(*) \quad \int_{\mu \in \mathbb{G}'} d\mu \{ N_{\Psi_\mu(j_n f), S_0}(r) \} \leq \int_{\mu \in \mathbb{G}'} d\mu \{ N_{W(f_m^\mu), S_0}(r) - T_{f, R_\mu}(r) \} +$
$\varepsilon T_{f,E}(r)$, where $\varepsilon T_{f,E}(r)$ is the admissible eror term.

We plug this inequality to (X) $\Rightarrow$ If K_X is big, the obstruction for hyperbolicity arises from the situation where the procedure defining $\int_{\mu \in \mathbb{G}'} d\mu \{ N_{W(f_m^\mu), S_0}(r) - T_{f, R_\mu}(r) \}$ does **NOT** make sense.

4-th Variation : Obstruction to Hyperbolicity (continued)

• Obstruction to the above argument defining the random variable $\{N_{W(f_m^\mu),S_0}(r) - T_{f,R_\mu}(r)\}_{\mu\in\mathbb{G}'}$ or equivalently $\{\Psi_\mu(j_n f)\}_{\mu\in\mathbb{G}'}$ does **NOT** make sense is classified.

(a) For $\forall\mu \in \mathbb{G}'$ the image $f_m^\mu(\mathbb{C}) \subset$ a proper linear subspace in $\mathbb{P}^n$. We cannot use Cartan/Ahlfors theory.

(b) the image $f(\mathbb{C}) \subset \mathrm{Bs}\{R_\mu\}_{\mu\in\mathbb{G}'}$, i.e., the situation characterized by the condition $f(\mathbb{C}) \subset \bigcap_{\mu\in\mathbb{G}'} R_\mu$. The locus $\bigcap_{\mu\in\mathbb{G}'} R_\mu$ is characterized by the property that the pull-back of affine coordinates don't constitute a coordinate system on X.

• In the proof of the key inequality $(*)$ [P. Lin and K], we need almost surely existence of local holomorphic coordinate system obtained by the pull-back of affine functions on $\mathbb{P}^n$. If $f(\mathbb{C}) \subset \bigcap_{\mu\in\mathbb{G}'} R_\mu$, it is absolutely impossible.

4-th Variation : Obstruction to Hyperbolicity (continued)

Note : In both possibilities (a) and (b) we have to take sufficiently large m in order to apply the measure concentration phenomenon.

- It turns out that the possibility (a) never occurs, if D is a s.n.c. divisor.

- The possibility (b) does happen. For instance, the generalized diagonal.

- To prove the key inequality $(*)$ we have to make m large (so that the measure concentration takes place) and fixed. So, the special set should be the union of $\bigcap_{\mu\in\mathbb{G}'} R_\mu$ over those m. This union is a proper algebraic subset. We fix a large number M (so that measure concentration takes place) and define the special set

$$Z_{X,D,\varepsilon,E} := \bigcup_{m\le M} \bigcap_{\mu\in\mathbb{G}'} R_\mu \ ,$$

where $\varepsilon > 0$ and E means that we admit an error of magnitude $\varepsilon T_{f,E}(r)$ in the key inequality $(*)$.

4-th Variation : Obstruction to Hyperbolicity (continued)

• (Example) Linear case : Imagine 4 lines in general position in $\mathbb{P}^2$. A line passing through 2 intersection points among 6 is a generalized diagonal. $X = \mathbb{P}^n$, D : linear divisor in general position. Then $\mathrm{Bs}\{R_\mu\}_{\mu\in\mathbb{G}}$ consists of the generalized diagonal. $\because$ Let S be a k-codimensional linear subspace of $\mathbb{P}^n$ s.t. $S \cap D$ has component-wise multiplicity $k+1$, i.e., S is a generalized diagonal. The restriction to an ε-displacement S_ε of S of a holomorphic section of $[mD]$ vanishing along D is locally expressed as $(x + a_1\varepsilon y_1)\ldots(x + a_k\varepsilon y_k)(x + a_{k+1}\varepsilon y_{k+1})$. We differentiate this quantity along S_ε in a direction transversal to S_ε. We can choose such a direction so that $(x + a_1\varepsilon y_1)\ldots(x + a_{k+1}\varepsilon y_{k+1}) - x^k = O(\varepsilon^2)$. If we choose this direction, the differentiation of this quantity w.r.to ε is of magnitude $O(\varepsilon)$. Therefore S belongs to any R_μ.

Thank you very much for your attention !