Generalized Ricci Flow

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One can now express the flow equivalently as

$$\frac{\partial}{\partial t}(g - b) = -2\,\mathrm{Rc}^{\nabla}, \qquad H = H_0 + db.$$

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Thus one might expect different behavior of the generalized Ricci flow on a given manifold depending on $[H]$.

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Generalized geometry

The foundational object in generalized geometry is the generalized tangent bundle, endowed with the neutral inner product and the Dorfman/Courant bracket:

$$E = TM \oplus T^*M, \qquad \langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}\left(\eta(X) + \xi(Y)\right)$$
$$[X + \xi, Y + \eta] = [X, Y] + L_X \eta - i_Y d\xi + i_Y i_X H,$$

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is a positive definite inner product on $T \oplus T^*$. This data is equivalent to a pair (g, b) of a Riemannian metric on two-form b such that

$$\mathcal{G} = e^{-b}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{b} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}$$

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Generalized geometry and generalized Ricci flow

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With the above point of view, it is possible to recast the generalized Ricci flow as a flow of generalized metrics:

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2\,\mathrm{Rc} + \tfrac{1}{2} H^2 \\ \frac{\partial}{\partial t} H &= \Delta_d H \end{aligned} \qquad \longleftrightarrow \qquad \mathcal{G}^{-1}\frac{\partial}{\partial t}\mathcal{G} = -2\mathcal{RC}.$$

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3. GRF is a gradient flow: Let

$$\mathcal{F}(g, H, f) = \int_M \left(R - \tfrac{1}{12} |H|^2 + |\nabla f|^2 \right) e^{-f} dV_g,$$
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Theorem

(Oliynik-Suneeta-Woolgar 2006) Generalized Ricci flow is the gradient flow of λ.

S^1 invariant solutions on three-manifolds

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It turns out that this ansatz is preserved by GRF, and moreover reduces to Ricci-Yang-Mills flow:

$$\begin{aligned} \frac{\partial}{\partial t} G &= -2 \operatorname{Rc}^G + \tfrac{1}{2} H^2 \\ \frac{\partial}{\partial t} H &= \Delta_d H \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} \frac{\partial}{\partial t} g &= -2 \operatorname{Rc}^g + F^2 \\ \frac{\partial}{\partial t} \mu &= -d_g^* F \end{aligned}$$

Theorem

(______ 2021) Let $S^1 \to M \to \Sigma$ *denote a circle bundle over a Riemann surface* Σ*. Let* (g_t, μ_t) *denote a solution to Ricci-Yang-Mills flow on* M*, with* $G_t = \pi^* g_t + \mu_t \otimes \mu_t$ *the associated one-parameter family of invariant metrics on* M*.*

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1. *If* $\chi(\Sigma) < 0$ *then* (g_t, μ_t) *exists on* $[0, \infty)$ *and* $(M, \frac{G_t}{2t})$ *converges to* (Σ, g_Σ)*, where* g_Σ *denotes the canonical metric of constant curvature* -1*.*

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2. *If $\chi(\Sigma) = 0$ then (g_t, μ_t) exists on $[0, \infty)$ and $(M, \frac{G_t}{2t})$ converges to a point.*
3. *If $\chi(\Sigma) > 0$ and $c_1(M) = 0$, then there exists $T < \infty$ such that (g_t, μ_t) exists on $[0, T)$, and $(M, \frac{1}{T-2t} G_t)$ converges to $(\Sigma \times \mathbb{R}, g_\Sigma \times g_\mathbb{R})$, where g_Σ denotes a metric of constant curvature 1.*

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4. *If $\chi(\Sigma) > 0$, and $c_1(M) \neq 0$, then (g_t, μ_t) exists on $[0, \infty)$ and converges to a quotient of the Bismut-flat S^3.*

Generalized Ricci flow on complex manifolds

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Definition

Given (M^{2n}, g, J) a Hermitian manifold, we say it is pluriclosed if

$$H := d^c\omega = \sqrt{-1}\left(\overline{\partial} - \partial\right)\omega, \qquad dH = 2\sqrt{-1}\partial\overline{\partial}\omega = 0.$$

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3. The local generality of pluriclosed metrics is that of a $(1,0)$-form, i.e. locally $\omega = \overline{\partial}\alpha + \partial\overline{\alpha}$.

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Theorem

(_____, Tian, 2010) Let (M^{2n}, ω_t, J) be a solution to pluriclosed flow. Let (g_t, H_t) be the associated 1-parameter families of Riemannian metrics and torsion forms. Then

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc} + \tfrac{1}{2}H^2 - L_{\theta^\sharp} g,$$
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What kind of global behavior can we expect on non-Kähler surfaces?

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(Gauduchon-Ivanov 1997) Let (M^4, g, J) *be a compact complex surface, where* g *is pluriclosed and satisfies*

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Furthermore, if ω_t is a solution of pluriclosed flow, then

$$G^{-1} \frac{\partial}{\partial t} G = -S^G.$$

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Fundamental structural results

Suppose (M^{2n}, g, J) is a Hermitian structure, $H = d^c\omega$, $dH = 0$, and

$$\mathrm{Rc} - \tfrac{1}{4}H^2 + \nabla^2 f = 0,$$
$$d^* H + i_{\nabla f} H = 0.$$

Proposition

With the hypotheses above, the vector field

$$V = \tfrac{1}{2}\left(\theta^\sharp - \nabla f\right)$$

is holomorphic, and JV is Killing.

Thus for a generalized Kähler-Ricci soliton, we obtain two Killing fields

$$IV_I = \tfrac{1}{2} I\left(\theta_I^\sharp - \nabla f\right), \qquad JV_J = \tfrac{1}{2} J\left(\theta_J^\sharp - \nabla f\right)$$

Thus either:

1. The isometry group has dimension at least two
2. The vectors IV_I and JV_J are aligned, thus there exists a biholomorphic Killing field.

Canonical local ansatz in nondegeneracy locus

Fix now (M^4, g, I, J) a generalized Kähler manifold with defined by a triple of symplectic forms $\Omega, I\Omega, J\Omega$, where

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Thus if M^4 admits a free, isometric, biholomorphic S^1 action, generated by X, we obtain local functions μ_i such that

$$d\mu_1 = i_X\Omega, \qquad d\mu_2 = i_X I\Omega, \qquad d\mu_3 = i_X J\Omega,$$

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This function p satisfies $|p| < 1$ and is constant if and only if the structure is hyperKähler.

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Crucially, this construction is reversible.

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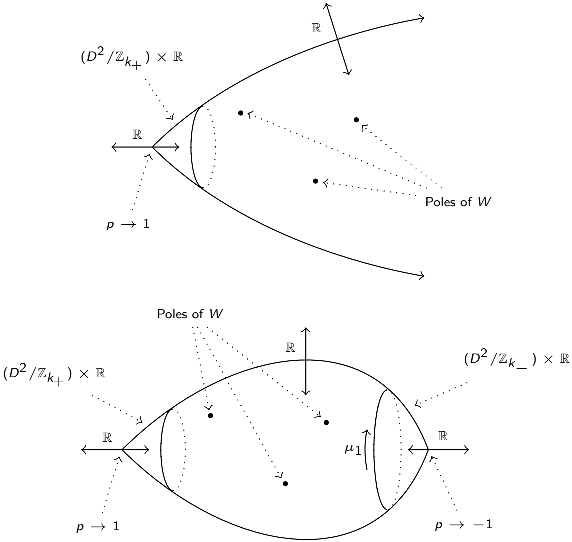
3. How can we produce and classify viable choices of W?

Generalized Gibbons-Hawking Ansatz

$$g = Wh + W^{-1}\eta^2, \qquad \Delta_h W + \dots = 0, \qquad d\eta = \star_h dW + \dots$$



Thank You!