# 多重標準線形系の $L^p$ 構造の極限について

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## Abstract

We study limits of the canonical $L^p$-space structures on pluricanonical systems on compact complex manifolds. We construct a canonical mixed $L^p$-space structure on the central fiber of a one parameter degeneration. We prove that the canonical $L^p$-spaces on smooth nearby fibers converge to the mixed $L^p$-space.

# §1. Introduction

$X$ a compact complex manifolds, $n$-dim.

Intrinsic objects: $\big(H^0(X, mK_X),\ \| \ \|\big)$.

$s \in H^0(X, mK_X)$ $m$-canonical form, $m \in \mathbb{Z}_{>0}$, put $p = 2/m \leq 2$.

$|s|^{2/m} = |s \wedge \overline{s}|^{1/m}$ a semi-positive volume form on $X$.

The $L^p$-pseudo-norm :

$$\|s\| := \left(\int_X |s|^p\right)^{1/p} = \left(\int_X |s \wedge \overline{s}|^{1/m}\right)^{m/2}.$$

$\|s\|$ defines an $L^2$-inner product if $m = 1$, an $L^1$-norm if $m = 2$,

merely an $L^p$-pseudo-norm if $m > 2$.

[NS], [BP], [PT], [HPS], ... : $m$-Bergman kernel metric, $m$-Narasimhan-Simha metric, variations of these metrics in a holomorphic family $\mathcal{X} \to S$, ...

We study the limit of $L^p$-space structures in a family, after Mabuchi.

Let $f : X \to C = \{|t| < 1\}$ be a proper surjective holomorphic, . . . ,

$f$ smooth over $C^*$, the central fiber $X_0$ is a reduced SNC divisor.

We have $L^p$-spaces $\left\{ \left( H^0(X_t, mK_{X_t}), \ \| \ \|_t \right) \right\}_{t \neq 0}$.

We can define a mixed $L^p$-space structure on

(the restricted linear subspace) $f_*(mK_{X/C})_0 \subset H^0(X_0, mK_{X_0})$.

**Theorem 1.1.** $\left( H^0(X_t, mK_{X_t}), \ \| \ \|_t \right) \longrightarrow \left( f_*(mK_{X/C})_0, \ \| \ \|_0 \right)$ *as* $t \to 0$ *in* $\mathcal{M}_{N,p}$ *with respect to the distance introduced by Mabuchi.*

Here, $\mathcal{M}_{N,p}$ : the space of all isomorphism classes of $L^p$-space structures of $\mathbb{C}^N$.

H. Mazur '76 : for a family of stable curves.

Mabuchi : for a family of canonically polarized manifolds.

Contents

## §2. The space of $L^p$-structures, Mabuchi's distance, . . .

$V \cong \mathbb{C}^N$. We take an euclidean metric $|\ \ | = |\ \ |_{eu}$. $p \in (0,2]$ (mostly $p \in (0,1]$).

**Def. 2.1.** $\|\ \ \| : V \to \mathbb{R}_{\geq 0}$ is an $L^p$-pseudo-norm, if

(1) $\|v\| = 0 \iff v = 0 \ \ (v \in V)$,

(2) $\|c\,v\| = |c|\|v\| \ \ (c \in \mathbb{C},\ v \in V)$,

(3) $\|u+v\|^p \leq \|u\|^p + \|v\|^p \ \ (u, v \in V)$. (Not $\|u+v\| \leq \|u\| + \|v\|$ in general.)

**Notation 2.2.** Let $(V, \|\ \ \|)$ be an $L^p$-space (of dimension $N$).

(1) The indicatrix (the unit ball) $\Sigma := \{v \in V;\ \|v\| \leq 1\}$.

(2) For a basis $\boldsymbol{v} = \{v_1, \dots, v_N\}$ of $V$, we let the associated ellipsoid

$$E(\boldsymbol{v}) = \{z \cdot \boldsymbol{v} = \textstyle\sum_{i=1}^N z_i v_i\ ;\ z \in \mathbb{C}^N,\ |z| \leq 1\}.$$

(3) Let $\mathcal{B}$ be the set of all bases $\boldsymbol{v}$ of $V$ such that $E(\boldsymbol{v}) \supset \Sigma$.

**Prop. 2.3.** *A special basis for $(V, \| \ \|)$ (substitute for ONB).*

*$\exists$ a basis $\boldsymbol{v} \in \mathcal{B}$, i.e. $E(\boldsymbol{v}) \supset \Sigma$, whose volume is minimum :*

$$\left| \frac{v_1 \wedge v_2 \wedge \ldots \wedge v_N}{w_1 \wedge w_2 \wedge \ldots \wedge w_N} \right| \leq 1$$

*for all $\boldsymbol{w} = \{w_1, \ldots, w_N\} \in \mathcal{B}$. It is unique up to the action of $U(N)$.*

**Def. 2.4.** Mabuchi's distance on $\mathcal{M}_{N,p}$ :

the set of isomorphism classes of all $L^p$-space structures of $\mathbb{C}^N$.

- For $\underline{V} = (V, \| \ \|) \in \mathcal{M}_{N,p}$ and a basis $\boldsymbol{v}$ for $V$, we let

$$f_{\boldsymbol{v}}(s) := \|s \cdot \boldsymbol{v}\| \ : \ S^{2N-1} \longrightarrow \mathbb{R}_{\geq 0}, \quad S^{2N-1} = \{s \in \mathbb{C}^N \ ; \ |s| = 1\}.$$

- Put $d : \mathcal{M}_{N,p} \times \mathcal{M}_{N,p} \to \mathbb{R}_{\geq 0}$ by

$$d\,(\underline{V}, \ \underline{V}') = \min_{\boldsymbol{v},\boldsymbol{v}'} \left\{ \ \|f_{\boldsymbol{v}} - f_{\boldsymbol{v}'}\|_{L^\infty(S^{2N-1})} \right\},$$

where $\boldsymbol{v}$ (resp. $\boldsymbol{v}'$) are special bases for $\underline{V}$ (resp. $\underline{V}'$).

**Theorem 2.5** (Mabuchi). *The set $\mathcal{M}_{N,p}$ is compact with respect to the topology induced from the distance $d$.*

## §3. Mixed $L^p$-space structure on $H^0(X, mK_X)$ for a SNC variety $X$

$X = \bigcup_{i=1}^r X_i$ a SNC variety, $n$-dim.

### 3.1. Poincaré residue, . . .

$a_0 : X^{[0]} := \amalg_{i=1}^r X_i \longrightarrow X$ the normalization.

$a_k : X^{[k]} := \amalg_{i_0<i_1<\ldots<i_k} \big(X_{i_0} \cap X_{i_1} \cap \ldots \cap X_{i_k}\big) \longrightarrow X$

the normalization of $(n-k)$-dim stratum of $X$.

$$X^{[k]+1} := a_k^{-1}\big(a_{k+1}(X^{[k+1]})\big) \text{ a SNC divisor on } X^{[k]}.$$

- The Poincaré residue map gives

$$\mathrm{Res}_k : H^0(X, mK_X) \longrightarrow H^0(X^{[k]}, m(K_{X^{[k]}} + X^{[k]+1})).$$

Letting $W_k = \mathrm{Res}_k^{-1}\big(H^0(X^{[k]}, mK_{X^{[k]}})\big) \subset H^0(X, mK_X)$,

we obtain a filtration

$$H^0(X^{[0]}, mK_{X^{[0]}}) \cong W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = H^0(X, mK_X).$$

$W_{-1} = \{0\}$ formally.

Note that

$$\mathrm{Res}_k : W_k \longrightarrow H^0(X^{[k]}, mK_{X^{[k]}}), \quad \mathrm{Res}_k(W_{k-1}) = 0, \quad \mathrm{Res}_{k+1}(W_k) = 0.$$

### 3.2. Mixed $L^p$-space structure.

Let $Gr_k = W_k/W_{k-1}$ with an injection $\mathrm{Res}_k : Gr_k \to H^0(X^{[k]}, mK_{X^{[k]}})$.

Here $H^0(X^{[k]}, mK_{X^{[k]}})$ admits the $L^p$-space structure $\| \ \|_k$. We put

$$\big(Gr\, H^0(X, mK_X), \| \ \|\big) := \bigoplus_{k=0}^{n} \big(Gr_k, \ \| \ \|_k\big)$$

the orthogonal direct sum, i.e. $\| \sum_k s_k \|^p = \sum_k \|s_k\|_k^p$.

- For a linear subspace $H \subset H^0(X, mK_X)$, one can restrict everything on $H$,

and obtain $$\big(Gr\, H, \| \ \|\big) = \bigoplus_{k=0}^{n} \big(Gr_{H,k}, \ \| \ \|_k\big).$$

E.g. If $X_0$ is a SNC variety, $X_0 \subset X$ is a fiber of $f : X \to C$,

$$H = H^0(X|X_0, mK_{X/C}) := \mathrm{Image}\big(H^0(X, mK_{X/C}) \longrightarrow H^0(X_0, mK_{X_0})\big)$$

the restricted linear subspace.

**Theorem 3.3.** *Let* $f : X \to C = \{|t| < 1\}$ *be a proper surjective, . . . ,*

$m \in \mathbb{Z}_{>0}$, $f_*(mK_{X/C})$ *is of rank* $N > 0$.

*(1) If* $X_0$ *is reduced SNC, then*

$$\big(H^0(X_t, mK_{X_t}), \|\ \|_t\big) \longrightarrow \big(Gr\, H^0(X|X_0, mK_{X/C}),\ \|\ \|\big)$$

*as* $t \to 0$ *in* $\mathcal{M}_{N,p}$.

*(2) Even if* $f$ *is not semi-stable,* $\exists!\ (V_0,\ \|\ \|_0) \in \mathcal{M}_{N,p}$ *s.t.*

$$\big(H^0(X_t, mK_{X_t}), \|\ \|_t\big) \longrightarrow (V_0,\ \|\ \|_0)$$

*as* $t \to 0$. *(After taking a semi-stable reduction, we can reduced to the situation (1).)*

*(3) Suppose* $f$ *is projective. If* $X_0$ *is normal and has canonical singularities at worst. Then the limit* $(V_0, \|\ \|_0)$ *in (2) is* $\big(H^0(X_0, mK_{X_0}), \|\ \|_0\big)$.

## §4. Asymptotic behavior of $L^p$-pseudo-norms / fiber integrals

Let $f : X \to C = \{|t| < 1\}$ ... semi-stable, $X_0 = F$ reduced SNC.

**Prop. 4.1.** *Let* $s \in H^0(X, mK_{X/C})$, *and* $s_t = s|_{X_t} \in H^0(X_t, mK_{X_t})$ *for* $t \in C$. *Suppose* $s_0 \in W_{H,k}$, *i.e.,* $\mathrm{Res}_k s_0 \in H^0(F^{[k]}, mK_{F^{[k]}})$. *Then,* $\exists\, \delta > 0, \exists\, A_s > 0$ *s.t. for* $\forall\, t \in C,\ 0 < |t| < \delta$,

$$\left| \|s_t\|_t^p - \Big( \int_{F^{[k]}} |\mathrm{Res}_k\, s_0|^p \Big) \frac{(2\pi)^k}{k!} (-\log|t|^2)^k \right| \leq A_s(-\log|t|)^{k-1}$$

*holds.*

**Uniform version 4.2.** *Let* $s_1, \ldots, s_\ell \in H^0(X, mK_{X/C})$, *and* $s_{1,0}, \ldots, s_{\ell,0} \in W_{H,k}$ *(we allow* $s_{i,0} \in W_{H,k-1}$*). Then,* $\exists\, \delta > 0, \exists\, A > 0$ *s.t. for* $\forall\, t \in C,\ 0 < |t| < \delta$, *and* $\forall\, z = (z_1, \ldots, z_\ell) \in \mathbb{C}^\ell,\ |z|_{eu} \leq 1$,

$$\left| \| \sum_i z_i s_{i,t} \|_t^p - \Big( \int_{F^{[k]}} |\mathrm{Res}_k \sum_i z_i s_{i,0}|^p \Big) \frac{(2\pi)^k}{k!} (-\log|t|^2)^k \right| \leq A(-\log|t|)^{k-1}$$

*holds.*

**Continuity of a normalized $L^p$-pseudo-norm.** $F = X_0$.

May suppose $f_*(mK_{X/C}) \cong \bigoplus_{i=1}^N \mathcal{O}_C\, s_i$

with $s_i \in H^0(C, f_*(mK_{X/C})) \cong H^0(X, mK_{X/C})$.

We'd like to show the continuity of $\big(f_*(mK_{X/C})_t, \|\ \|_t\big)$ at $t = 0$.

This is very much related to the continuity of each $\|s_{i,t}\|_t$ at $t = 0$

(after an appropriate normalization).

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Let $s \in H^0(X, mK_{X/C})$. Consider $k = k_s$ the minimum $k$ s.t. $s_0 \in W_{H,k}$.

$$\underline{s} = \frac{s}{L_k^{1/p}}, \quad L_k (= L_k(t)) = \frac{(2\pi)^k}{k!}(-\log|t|^2)^k \quad \text{for } t \neq 0, \text{ or on } X \setminus X_0.$$

We may call $\underline{s}$ a normalization of $s$. We put

$$\underline{s}_0 := \mathrm{Res}_k\, s_0 \in H^0(F^{[k]}, mK_{F^{[k]}}).$$

Then, Proposition 4.1 and 4.2 conclude

**Cor. 4.3.** *$\exists\,\delta > 0$, $\exists\,A_s > 0$ depending only on the "$C^0$-norm" of $s$ on a neighborhood of $X_0 = F$ s.t. for $\forall\, t \in C$, $0 < |t| < \delta$,*

$$\Big|\; \|\underline{s}_t\|_t^p - \|\underline{s}_0\|_0^p \;\Big| \leq A_s(-\log|t|)^{-1} \quad (\to 0 \text{ as } t \to 0)$$

*holds. There is also a uniform version.*

We actually need a uniform continuity to conclude the continuity of $L^p$-spaces $\big(f_*(mK_{X/C})_t, \|\ \|_t\big)$ at $t = 0$.

**Cor. 4.4. Convergence as currents.** (the minimum $k$ s.t. $s_0 \in W_{H,k}$)
*The family of distributions $\{\sigma_t\}_{t \in C \setminus 0}$ on $X$ given by*

$$\sigma_t \;:\; C^0_{\mathrm{cpt}}(X, \mathbb{C}) \ni \varphi \;\mapsto\; \int_{X_t} \varphi\, |\underline{s}_t|^p,$$

*converges to a Dirac type distribution*

$$\sigma_0 \;:\; \varphi \;\mapsto\; \int_{F^{[k]}} (a_k^* \varphi)\, |\mathrm{Res}_k\, s_0|^p$$

*on $X$ supported on $a_k(F^{[k]})$.*

**Prop. 4.5. Asymptotic orthogonality.**

*Let* $s^{(k)} \in H^0(X, mK_{X/C})$ *s.t.* $s_0^{(k)} \in W_{H,k} \setminus W_{H,k-1}$,

$s^{(\ell)} \in H^0(X, mK_{X/C})$ *s.t.* $s_0^{(\ell)} \in W_{H,\ell} \setminus W_{H,\ell-1}$. *Suppose* $k \neq \ell$.

*Then, as* $t \to 0$*, we have*

$$\|\underline{s}_t^{(k)} + \underline{s}_t^{(\ell)}\|_t^p = \|\underline{s}_t^{(k)}\|_t^p + \|\underline{s}_t^{(\ell)}\|_t^p + O\big((-\log|t|)^{-1/m}\big).$$

*There is also a uniform version.*

*Proof.* By Cor 4.4, $\operatorname{Supp} \sigma_0^{(k)} = a_k(F^{[k]})$, $\operatorname{Supp} \sigma_0^{(k+1)} = a_{k+1}(F^{[k+1]})$.

$$\sigma_0^{(k)} = 0 \quad \text{along} \quad a_{k+1}(F^{[k+1]})$$

by definition of $k$ : the minimum s.t. $s_0 \in W_{H,k}$. □

This is a key property in various situations, e.g. the Mumford goodness of the canonical $L^2$-metric $g$ on $f_*(K_{X/Y} \otimes L)$.