Weighted K-stability and Extremal Sasaki metrics

Vestislav Apostolov

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Plan of the lecture

Based on joint works

- D. J. Calderbank, A., arXiv:1810.10618.
- D. J. Calderbank, E. Legendre, A., arXiv:2012.08628
- S. Jubert, A. Lahdili, A., arXiv:2104.09709

- Extremal Sasaki structures via weighted extremal Kähler metrics
- The weighted Calabi problem and weighted K-stability
- Discussion of proofs
- Applications

Sasaki structures

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(1) a $2n$-dimensional distribution $\mathcal{D} \subset TN$ with a point-wise complex structure $J_x : \mathcal{D}_x \to \mathcal{D}_x$ such that

$$[\mathcal{D}^{1,0}, \mathcal{D}^{1,0}] \subset \mathcal{D}^{1,0},$$

$(\mathcal{D}, J)$ is called a CR structure on N.

Sasaki structures

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A **Sasaki structure** on a (connected) $(2n+1)$-manifold N is:

(1) $(\mathcal{D}, J)$ a CR structure;

(2) $(\mathcal{D}, J)$ is strictly pseudo-convex, i.e. its Levi form

$$L_{\mathcal{D}} : \wedge^2\mathcal{D}^* \to TN/\mathcal{D}, \qquad L_{\mathcal{D}}(X, Y) = -[X, Y] \mod \mathcal{D}$$

is a strictly definite $(1,1)$-form on $(\mathcal{D}, J)$.

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(2) s.t. $(\mathcal{D}, J)$ is strictly pseudo-convex;

(3) a *Sasaki–Reeb vector field* $\xi \in C^\infty(N, TN)$. i.e.

$$\mathcal{L}_\xi \mathcal{D} \subset \mathcal{D}, \qquad \mathcal{L}_\xi J = 0,$$

$$[\xi] \in C^\infty(N, TN/\mathcal{D}) \text{ does not vanish}, \qquad \omega_\xi := L_{\mathcal{D}}/[\xi] > 0.$$

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$(\xi, \mathcal{D}, J) \Leftrightarrow (\mathcal{D}, J, \omega_\xi, g_\xi)$ a ξ-transversal Kähler structure on N.

Regular Sasaki structures

A basic example

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- $TN = \mathbb{R} \cdot \chi \oplus_\eta \mathcal{D}$ where $\chi \in C^\infty(N, TN)$ is the generator of the $\mathbb{S}^1$-action $\Rightarrow$ lift J and ω to $\mathcal{D}$
- $(\chi, \mathcal{D}, J)$ is a regular Sasaki structure on N with ω_χ the lifted Kähler structure ω from M.

General Principle/Slogan

The regular Sasaki construction holds locally, around each point $x \in N$, and allows one to extend geometric notions from the space of local orbits $(M_\xi, J_\xi, \omega_\xi)$ of the flow of ξ (irrespective of the regularity of ξ) to corresponding notions on $(\xi, \mathcal{D}, J)$.

Geometric notions on Sasaki manifolds

Definition (Boyer–Galicki–Simanca)

A Sasaki structure $(\xi, \mathcal{D}, J)$ on N is

- Sasaki–Einstein if $(M_\xi, J_\xi, \omega_\xi)$ is Kähler-Einstein;
- CSC if the scalar curvature $Scal_\xi$ of $(M_\xi, J_\xi, \omega_\xi)$ is constant;
- extremal if $(M_\xi, J_\xi, \omega_\xi)$ is extremal, i.e. $\mathrm{grad}_{\omega_\xi}(Scal_{\omega_\xi})$ is Killing.

Why bother?

Facts

- (Kobayashi) if (M, J, ω) KE Fano ($L = K_M^*$) $K_M^\times := K_M \setminus O_M$ has structure of an affine variety in $\mathbb{C}^N$, with an isolated singularity at 0, which admits a Calabi–Yau "cone" Kähler metric.

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- (Martelli–Sparks–Yau) More generally (irregular) positive Sasaki–Einstein structures give rise to CY affine cones.
- (Collins–Szekelyhidi) positive CSC Sasaki structures give rise to scalar-flat Kähler metrics on affine cones and $\exists$ obstructions.

Sasaki–Reeb fields versus Killing potentials

Consider $(N, \chi, \mathcal{D}, J) \to (M, J, g, \omega)$ regular and suppose $(\xi, \mathcal{D}, J)$ is another Sasaki structure with $[\xi, \chi] = 0$.

Sasaki–Reeb fields versus Killing potentials

Consider $(N, \chi, \mathcal{D}, J) \to (M, J, g, \omega)$ regular and suppose $(\xi, \mathcal{D}, J)$ is another Sasaki structure with $[\xi, \chi] = 0$.

$$[\xi] = f[\chi] \in TN/\mathcal{D}, \qquad f \in C^\infty(N)^\chi, \qquad f > 0.$$

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Lemma 1

- f descends to a positive function on M such that $\check{\xi} := J\,\mathrm{grad}_g f$ is a Killing vector field.
- any positive Killing potential $f > 0$ on (M, J, g, ω) defines a Sasaki structure on $(N, \mathcal{D}, J)$ by

$$\xi := f\chi - (\omega_\chi)^{-1}(df)_{\mathcal{D}}.$$

Extremal Sasaki versus weighed extremal Kähler

$(N, \chi, \mathcal{D}, J) \to (M, J, g, \omega)$ regular Sasaki; $(\xi, \mathcal{D}, J)$ Sasaki with $[\xi, \chi] = 0$; $f := f_\xi > 0$ the Killing potential (Lemma 1):

Lemma 2 (Calderbank–A.; Jubert–Lahdili–A.)

- $(\xi, \mathcal{D}, J)$ is extremal Sasaki iff

  $$Scal_f(g) := f^2 Scal(g) - 2(n+1) f \Delta_g f - (n+2)(n+1)|\nabla^g f|_g^2$$

  is a Killing potential ($Scal_f$-extremal).
- $(\xi, \mathcal{D}, J)$ is CSC iff $Scal_f(g) = cf$, $c \in \mathbb{R}$.
- $(\xi, \mathcal{D}, J)$ is Sasaki–Einstein iff $Ric(g) - \lambda\omega = -\frac{(n+2)}{2} dd^c \log f$.

Proof of Lemma 2

$$TN = \mathbb{R}\cdot\chi \oplus \mathcal{D} = \mathbb{R}\cdot\xi \oplus \mathcal{D};$$

$$(\eta^{\chi})_{\mathcal{D}} = 0,\ \eta^{\chi}(\chi) = 1, \qquad (\eta^{\xi})_{\mathcal{D}} = 0,\ \eta^{\xi}(\xi) = 1.$$

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- $Scal(g_\xi)$ of $(\xi, \mathcal{D}, J) \Leftrightarrow$ Tanaka–Webster scalar curvature of $(\eta_\xi, \mathcal{D}, J)$ (conformal transformation of Tanaka–Webster curvature, see e.g. Jerison–Lee)

$$\begin{aligned} Scal(g_\xi) &= f\,Scal(g_\chi) - 2(n+1)\Delta_{g_\chi} f - \frac{(n+2)(n+1)}{f}|df|^2_{g_\chi} \\ &= \frac{1}{f} Scal_f(g_\chi). \end{aligned}$$

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$$Scal(g_\xi) = \frac{1}{f} Scal_f(g_\chi), \qquad (*)$$

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$$V := Scal(g_\xi)\xi - \omega_\xi^{-1}\left(dScal(g_\xi)\right)_{\mathcal{D}}.$$

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Using $(*)$, $\xi = f\chi - \omega_\chi^{-1}(df_{\mathcal{D}})$ and $\omega_\xi = \frac{1}{f}\omega_\chi$:

$$V = Scal_f(g_\chi)\chi - \omega_\chi^{-1}(dScal_f(g_\chi))_{\mathcal{D}}.$$

Calabi problem

Problem (Calabi problem)

Given (M, J, ω_0), find a deformation

$$K_{\omega_0}(M, J) = \{\omega_\varphi = \omega_0 + dd^c\varphi > 0, \ \varphi \in C^\infty(M)\}$$

within the cohomology class $[\omega] \in H^2_{dR}(M)$ such that $(g_\varphi, \omega_\varphi)$ is extremal Kähler, i.e. $J \,\mathrm{grad}_{g_\varphi}(Scal(g_\varphi))$ is Killing.
$Scal(g_\varphi) = const$ is the CSC problem and
$Ric(g_\varphi) = \frac{Scal(g_\varphi)}{2n}\omega_\varphi$ is the Kähler–Einstein problem.

A weighted Calabi problem

- $(N, \mathcal{D}, J, \chi) \xrightarrow{p} (M, J, \omega_0)$ regular Sasaki;
- $(\xi, \mathcal{D}, J)$ another Sasaki structure with $[\xi, \chi] = 0$;
- $\mathbb{T} \subset \mathrm{Aut}_r(M, J)$ generated by $\check{\xi} := J \,\mathrm{grad}_g f$.

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$$K_{\omega_0}(M, J)^{\mathbb{T}} = \{\omega_\varphi = \omega_0 + dd^c\varphi > 0,\ \varphi \in (C^\infty(M))^{\mathbb{T}}\}$$

ω_φ defines a new connection 1-form

$$\eta_\varphi = \eta_0 + p^*(d^c\varphi)$$

and Sasaki structure $(N, \chi, \mathcal{D}_\varphi, J_\varphi)$ on N.

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Fact: $(\xi, \mathcal{D}_\varphi, J_\varphi)$ Sasaki with induced Killing potential

$$f_\varphi = \eta_\varphi(\xi) = f + (d^c\varphi)(\check{\xi}).$$

A weighted Calabi problem

(Calderbank–Legendre–A.)

Existence of extremal Sasaki structure with Reeb field ξ on $p : N \to M \Leftrightarrow$

Problem (The weighted Calabi problem)

Find $\omega_\varphi \in K_{\omega_0}(M, J)^{\mathbb{T}}$ *s.t.*

$$\begin{aligned} Scal_{f_\varphi}(g_\varphi) =& f_\varphi^2 Scal(g_\varphi) - 2(n+1) f_\varphi \Delta_{g_\varphi} f_\varphi \\ & - (n+2)(n+1)|df_\varphi|^2_{g_\varphi} \end{aligned}$$

is a Killing potential, where $f_\varphi := f + d^c\varphi(\check{\xi})$ *is the Killing potential of* $\check{\xi} \in \mathrm{Lie}(\mathbb{T})$ *with respect to* g_φ*.*

Other weighted cscK problems

(after Lahdili)

More general setting: $\mathbb{T}$-invariant (M, J, ω_0), $\mathbb{T} \subset \mathrm{Aut}_r(M, J)$, $\omega_\varphi \in K_{\omega_0}(M, J)^{\mathbb{T}}$:

$$m_\varphi := m_0 + d^c\varphi, \qquad m_\varphi(M) = m_0(M) = P \subset (\mathrm{Lie}(\mathbb{T}))^*,$$

be the normalized momentum maps and $v(x), w(x)$ be smooth functions (with $v(x) > 0$). Then we introduce

$$Scal_v(g_\varphi) := v(m_\varphi)Scal(g_\varphi) + 2\Delta_{g_\varphi} v(m_\varphi) + \sum_{i,j=1}^{\ell} v_{,ij}(m_\varphi) g_\varphi(\xi_i, \xi_j)$$

and study the PDE for $\varphi \in K_{\omega_0}(M, J)^{\mathbb{T}}$:

$$Scal_v(g_\varphi) = w(m_\varphi). \qquad (**)$$

Other weighted cscK problems

(after Lahdili)

Definition (Lahdlili)

A solution $(g_\varphi, \omega_\varphi) \in K_{\omega_0}(M, J)^{\mathbb{T}}$ of

$$Scal_v(g_\varphi) = w(m_\varphi), \qquad (**)$$

where

$$Scal_v(g_\varphi) := v(m_\varphi)Scal(g_\varphi) + 2\Delta_{g_\varphi} v(m_\varphi) + \sum_{i,j=1}^{\ell} v_{,ij}(m_\varphi) g_\varphi(\xi_i, \xi_j)$$

and $m_\varphi := m_\omega + d^c\varphi$ is called a (v, w)-**cscK metric**.

Other weighted cscK problems

(after Lahdili)

Examples ((v, w)-cscK)

- $v = 1, w = const \Leftrightarrow$ cscK;
- $v = 1$, w is affine-linear $\Leftrightarrow$ Calabi extremal;
- $v = (\langle \check{\xi}, x\rangle + c)^{-(n+1)}, w = \ell_{\rm ext}(x)(\langle \check{\xi}, x\rangle + c)^{-(n+3)}$ with $\ell_{\rm ext}$ affine-linear $\Leftrightarrow$ weighted extremal Sasaki.
- $v = e^{\langle \check{\xi}, x\rangle}, w = (\langle \check{\xi}, x\rangle + a)e^{\langle \check{\xi}, x\rangle} \Leftrightarrow \mu$-cscK (E. Inoue).

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Remarks

For all these examples $w(x) = \ell_{\rm ext}(x)w_0(x)$ where $\ell_{\rm ext}(x)$ is affine-linear and $w_0 > 0$: (v, w)-**extremal Kähler**.

Other weighted cscK problems

(after Han–Li, Berman–Witt-Nystrom)

Definition (Han–Li)

Suppose (M, J) is Fano and $\omega \in 2\pi c_1(M)$. A v-**soliton** is $(g_\varphi, \omega_\varphi) \in K_{\omega_0}(M, J)^{\mathbb{T}}$ such that

$$Ric(g_\varphi) - \omega_\varphi = \frac{1}{2} dd^c \log v(m_\varphi).$$

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Examples

- $v(x) = e^{\langle \check{\xi}, x \rangle} \Leftrightarrow$ Kähler–Ricci soliton (Tian–Zhu);
- $v(x) = \left(\langle \check{\xi}, x \rangle + c\right)^{-(n+2)} \Leftrightarrow$ Sasaki–Einstein structure $(\xi, \mathcal{D}, J)$ on $p : N \to M$.
- v-soliton $\Leftrightarrow$ M Fano and $(v, \tilde{v})$-cscK with $\tilde{v} = 2\left(n + \sum_{i=1}^{r} \frac{v_{,i}(x) x_i}{v(x)}\right) v(x)$.

Summary

Main Conclusion

- *extremal Sasaki $\subset$ (v, w)-extremal Kähler manifolds/orbifolds.*
- *Sasaki–Einstein $\subset$ v-solitons on Fano manifolds/orbifolds.*

Weighted K-stability

(following Dervan–Ross, Sjöström-Dyrefelt, Lahdili)

Definition (Equivariant test configurations)

Let $(M, \omega, \mathbb{T})$ be a $\mathbb{T}$-invariant Kähler n-mfd. A **smooth $\mathbb{T}$-equivariant Kähler test configuration** is a Kähler $(n+1)$-mfd $(\mathscr{M}, \Omega)$, endowed with an isometric action of $\mathbb{T}$ and an additional holomorphic $\mathbb{C}^*$-action ρ, such that

- $\exists\ \pi : \mathscr{M} \twoheadrightarrow \mathbb{P}^1$ (equivariant with respect to ρ and ρ_0 on $\mathbb{P}^1$);
- $\mathbb{T}$ preserves $(M_x := \pi^{-1}(x), \omega_x := \Omega_{|M_x})$ and for $x \neq 0 \in \mathbb{P}^1$, $(M_x, [\omega_x])$ is $\mathbb{T}$-equivariant isomorphic to $(M, [\omega])$;
- $\mathscr{M} \setminus M_0 \cong (\mathbb{P} \setminus \{0\}) \times M$ ($\mathbb{C}^* \times \mathbb{T}_{\mathbb{C}}$-equivariantly).

Weighted K-stability

(following Lahdili)

Definition ((v, w)-weighted Donaldson–Futaki invariant)

Let $(\mathscr{M}, \Omega, \mathbb{T})$ be a $\mathbb{T}$-equivariant smooth test configuration of $(M, \omega, \mathbb{T})$. The (v, w)-weighted Donaldson–Futaki invariant is

$$\mathbf{DF}_{v,w}(\mathscr{M}, \Omega) := -\int_{\mathscr{M}} (Scal_v(\Omega) - w(m_\Omega)) \frac{\Omega^{n+1}}{(n+1)!} + (8\pi) \int_M v(m_\omega) \frac{\omega^n}{n!}.$$

Does not depend on the choice $\Omega \in [\Omega]$ and $\omega \in [\omega]$!

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Does not depend on the choice $\Omega \in [\Omega]$ and $\omega \in [\omega]$!

For $v = 1, w = \underline{s} := n\frac{c_1(M)\cdot[\omega]^{n-1}}{[\omega]^n}$ (Odaka, Wang):

$$\mathbf{DF}_{1,\underline{s}}(\mathscr{M}, \Omega) = C\left(\frac{\underline{s}}{n+1}[\Omega]^{n+1} + \mathscr{K}_{\mathscr{M}/\mathbb{P}^1} \cdot [\Omega]^n\right)$$

Weighted K-stability

Theorem (Lahdili-2019, weighted K-semistability)

Suppose $(M, \omega, \mathbb{T})$ *admits a* $\mathbb{T}$*-invariant* (v, w)*-extremal Kähler metric* $\omega_e \in [\omega]$ *and* $\mathbb{T} \subset \mathrm{Aut}_r(M)$ *is maximal. Then, for any* $\mathbb{T}$*-equivariant smooth test configuration* $(\mathscr{M}, \Omega)$ *with reduced central fibre* M_0*:*

$$\mathbf{DF}_{v,w}(\mathscr{M}, \Omega) \geq 0.$$

Weighted K-stability

Theorem (Lahdili-2019, weighted K-semistability)

Suppose $(M, \omega, \mathbb{T})$ admits a $\mathbb{T}$-invariant (v, w)-extremal Kähler metric $\omega_e \in [\omega]$ and $\mathbb{T} \subset \mathrm{Aut}_r(M)$ is maximal. Then, for any $\mathbb{T}$-equivariant smooth test configuration $(\mathscr{M}, \Omega)$ with reduced central fibre M_0:

$$\mathbf{DF}_{v,w}(\mathscr{M}, \Omega) \geq 0.$$

Main Theorem (Jubert-Lahdili-A., weighted K-polystability)

Suppose, moreover, $\mathbf{DF}_{v,w}(\mathscr{M}, \Omega) = 0$. Then $\mathscr{M} \setminus M_\infty \cong M \times (\mathbb{P}^1 \setminus \{\infty\})$ (product test configuration).

Weighted K-stability

Remarks

- $v = 1, w = \underline{s}$ (cscK metrics) proved by Berman–Darvas–Lu and Sjöström-Dyrefeld;
- extremal Sasaki case (for projective test configurations) by Calderbank–Legendre–A. using He–Li .
- v-solitons: stronger iff result by Han–Li.

Sketch of the proof

(Tian's properness principle)

Uses Lahdili's version of weighted Mabuchi energy

$$\mathbf{M}_{v,w} : K_{\omega_0}(M,J)^{\mathbb{T}} \to \mathbb{R}, \qquad \mathbf{M}_{v,w}(\omega_0) = 0,$$

$$(d_\varphi \mathbf{M}_{v,w})(\dot\varphi) = -\int_M \dot\varphi \Big(Scal_v(g_\varphi) - w(m_\varphi) \Big) \omega_\varphi^n.$$

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- $v = 1, w = \underline{s}$: $\mathbf{M}_{1,\underline{s}}$ is the Mabuchi energy;
- (Lahdili) (v,w)-extremal $\Rightarrow$ global minimum of $\mathbf{M}_{v,w}$;
- (Lahdili) $(\mathscr{M}, \Omega, \mathbb{T}) \Rightarrow$ a ray $\omega_\tau := \Omega_{|M_x}$ in $K_{\omega_0}(M,J)^{\mathbb{T}}$ $(x := e^{(-\tau+\sqrt{-1}\theta)} \in \mathbb{C}^* \subset \mathbb{P}^1)$; if M_0 is reduced

$$\lim_{\tau\to\infty} \frac{\mathbf{M}_{v,w}(\omega_\tau)}{\tau} = \mathbf{DF}_{v,w}(\mathscr{M}, \Omega).$$

- $\mathbf{M}_{v,w}$ **bounded below** $\Rightarrow \mathbf{DF}_{v,w}(\mathscr{M}, \Omega) \geq 0$.

Sketch of the proof

Tian's properness principle

To further improve, we consider properness:

Definition (Tian, Darvas–Rubinstein)

$\mathbb{G} \subset \mathrm{Aut}_r(M,J)$ reductive. $\mathbf{M}_{v,w}$ is $\mathbb{G}$-**coercive** if $\exists\ \lambda, \delta > 0$:

$$\mathbf{M}_{v,w}(\omega_\varphi) \geq \lambda \inf_{g \in \mathbb{G}} \mathbf{J}(g^*\omega_\varphi) - \delta, \qquad \forall\, \omega_\varphi \in K_{\omega_0}(M,J)^{\mathbb{T}},$$

where $\mathbf{J}$ is a certain "norm" (called J-functional) on $K_{\omega_0}(M,J) \subset C^\infty(M)/\mathbb{R}$.

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Theorem (Tian, Berman–Darvas–Lu, Dervan–Dyrefelt, Hisamoto, Chi Li)

Suppose $\mathbb{G} = \mathbb{T}_{\mathbb{C}}$ with $\mathbb{T}$ a maximal torus in $\mathrm{Aut}_r(M, J)$ and $\mathbf{M}_{v,w}$ is $\mathbb{G}$-coercive. Then $\mathbf{DF}_{v,w}(\mathscr{M}, \Omega) = 0$ iff $\mathscr{M}$ is a product test configuration.

Sketch of the proof

Main argument is to show (v, w)-extremal $\Rightarrow$ $\mathbf{M}_{v,w}$ is $\mathbb{T}_{\mathbb{C}}$-coercive.

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- (Darvas–Rubinstein) axiomatic approach to

$\exists$ critical point of $\mathbf{M}$ $\Rightarrow$ $\mathbf{M}$ is $\mathbb{G}$-coercive.

In our case, need to

- Extend $\mathbf{M}_{v,w}$ to the metric completion $\mathcal{E}^1_\omega(M, J)^{\mathbb{T}}$ of $K_{\omega_0}(M, J)^{\mathbb{T}}$ w.r.t. $d_1(\varphi_0, \varphi_2) = \inf_{\varphi(t)} \int_0^1 |\dot{\varphi}(t)|(\omega_{\varphi(t)})^n$.
- Show weak compactness of $\mathbf{M}_{v,w}$.
- (Lahdili) $\mathbf{M}_{v,w}$ is convex along $C^{1,\bar{1}}$-geodesics in $\mathcal{E}^1_\omega(M, J)^{\mathbb{T}}$ (uses adaptation of Berman–Berndtsson);
- Show $\mathbb{G} = \mathbb{T}_{\mathbb{C}}$ acts transitively on the set of minima on $\mathbf{M}_{v,w}$ (uses adaptation of Berman–Darvas–Lu);

The key argument

Define the greatest LSC extension of $\mathbf{M}_{v,w}$ to $\mathcal{E}^1_\omega(M,J)^{\mathbb{T}}$.

$$\mathbf{M}_{v,w}(\omega_\varphi) = \int_M \log\left(\frac{\omega_\varphi^n}{\omega^n}\right) v(m_\varphi)\omega_\varphi^n - 2\mathbf{I}_v^{\mathrm{Ric}(\omega)}(\varphi) + \mathbf{I}_w(\varphi),$$

$$(d_\varphi \mathbf{I}_w)(\dot\varphi) := \int_M \dot\varphi w(m_\varphi)\omega_\varphi^n$$

$$(d_\varphi \mathbf{I}_v^{\mathrm{Ric}(\omega)})(\dot\varphi) := n\int_M \dot\varphi\Big(v(m_\varphi)\mathrm{Ric}(\omega)\wedge\omega_\varphi + \langle dv(m_\varphi), m_{\mathrm{Ric}(\omega)}\rangle\Big)\omega_\varphi^n.$$

$$m_\varphi = m_\omega + d^c\varphi.$$

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(after Han-Li, see also Donaldson)

Uses idea from a recent work of Han–Li on v-solitons:

- v-soliton (M is Fano) $\Leftrightarrow$ $(v, \tilde{v})$-cscK with

$$\tilde{v} = 2\left(n + \sum_{i=1}^{r} \frac{v_{,i}(x)x_i}{v(x)}\right) v(x),$$

and $\mathbf{M}_v^{\mathrm{HL}} = \mathbf{D}_v^{\mathrm{HL}} + C = \mathbf{M}_{v,\tilde{v}} + C_1$.

The key argument

(after Han-Li, see also Donaldson)

- $P \to B := B_1 \times \cdots \times B_k$ principal $\mathbb{T}$-bundle over cscK (B_i, ω_{B_i});
- $\theta \in \Omega^1(P, \mathfrak{t})$ connection: $d\theta = \sum_{j=1}^{k} \omega_{B_j} \otimes p_j$, $p_j \in \mathfrak{t}$.
- $\pi : Y := X \times_{\mathbb{T}} P \to B$ a principal X-bundle.
- bundle-like metrics on Y:

$$\omega^Y_\varphi = \omega^X_\varphi + \sum_{j=1}^{n} (\langle p_j, m_\varphi \rangle + c_j)\pi^* \omega_{B_j} + \langle dm_\varphi \wedge \theta \rangle.$$

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$$\omega_\varphi^Y = \omega_\varphi^X + \sum_{j=1}^n (\langle p_j, m_\varphi \rangle + c_j)\pi^*\omega_{B_j} + \langle dm_\varphi \wedge \theta \rangle.$$

Lemma 3 (Jubert–Lahdili–A.)

- $Scal(g_\varphi^Y) = Scal_p(g_\varphi^X) + q(m_\varphi)$ with $p(x) := \prod_{j=1}^d (\langle p_j, x\rangle + c_j)^{\dim(B_j)}$ and $q(x) := \cdots$.
- $\mathbf{M}^Y_{1,\bar{s}}(\omega_\varphi^Y) = C\mathbf{M}^X_{p,w}(\omega_\varphi^X)$.

An application to Sasaki geometry

Corollary (Jubert-Lahdili-A. + Han-Li)

Suppose $X \hookrightarrow Y \to B$ Fano with X toric Fano. Then $K_Y^{\times}$ admits a Calabi–Yau cone metric.

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Remarks

Extends a result by Futaki–Ono–Wang from toric Fano to principle toric Fano fibrations and Mabuchi's result from $\mathbb{P}^1$-bundles to more general fibrations.

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THANK YOU !