# Degeneration of Riemann surfaces and small eigenvalues of Laplacian

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# A theorem of Schoen-Wolpert-Yau

Let us recall some classical theorems concerning small eigenvalues of Laplacian acting on functions on a compact Riemann surface.

Let $M$ be a compact Riemann surface of genus $g > 1$ endowed with a Riemannian metric.

Let $\mathcal{C}$ be the disjoint union of simple closed geodesics of $M$ such that $M \setminus C$ consists of $n+1$ components. Let $\mathcal{C}_n$ be the set of all those $C$. For $C \in \mathcal{C}_n$, write $L(C)$ for the length of $C$. Set

$$\ell_n := \inf\{L(C);\, C \in \mathcal{C}_n\}.$$

Let

$$0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_n \le \cdots$$

be the eigenvalues of the Laplacian acting on the functions on $M$.

### Theorem (Schoen-Wolpert-Yau)

*Let $k > 0$ be a constant. Assume that the Gauss curvature $K$ satisfies*

$$-1 \le K \le -k.$$

*Then $\exists$ positive constants $\alpha_1, \alpha_2 > 0$ depending only on $g$ s.t.*

$$\alpha_1 k^{3/2} \ell_n \le \lambda_n \le \alpha_2 \ell_n \qquad (1 \le n \le 2g-3)$$

$$\alpha_1 k \le \lambda_{2g-2} \le \alpha_2.$$

## Theorem (Mazur)

*Let $f: X \to \Delta$ be a degenerating family of hyperbolic Riemann surfaces such that $X_0 = f^{-1}(0)$ is a stable curve. Let $\gamma_s^{\text{hyp}}$ be the hyperbolic metric on the fiber $X_s = f^{-1}(s)$ with $K_s = -1$. Let $p \in X_0$ be a node such that $f(z,w) = zw$ around $p$. Then $\exists$ constants $C_1, C_2 > 0$ such that*

$$C_1 \frac{dz d\bar{z}}{|z|^2 (\log |z|)^2} \leq \gamma_s^{\text{hyp}} \leq C_2 \frac{dz d\bar{z}}{|z|^2 (\log |z|)^2}$$

*on the domain $\{(z,w) \in X_s;\ |z| < 1,\ |w| < 1\} \cong \{z \in \Delta;\ |s| < |z| < 1\}$.*

The length of the pinched geodesic corresponding to $p \in \operatorname{Sing} X_0$ behaves asymptotically like $1/\log(1/|s|)$. From this estimate and the Schoen-Wolpert-Yau theorem, for degenerations to stable curves, we have:

$$\lambda_i(s) \sim \frac{1}{\log(1/|s|)} \qquad (i < \#\{\text{irreducible components of } X_0\}).$$

# Cheeger's inequality for the first eigenvalue

Recall that the Cheeger constant $h(M)$ is defined as

$$h(M) := \inf_{M \setminus N = M_1 \cup M_2, \partial M_i = N} \frac{L(N)}{\min\{A(M_1), A(M_2)\}}.$$

Here $N$ is a smooth curve of $M$ such that $M \setminus N = M_1 \cup M_2$ consists of two components, $L(N)$ is the length of $N$, $A(M_i)$ is the area of $M_i$.

**Theorem (Cheeger)**

$$\lambda_1 \geq \frac{h(M)^2}{4}.$$

For a degenerating family of hyperbolic Riemann surfaces to a stable curve,

$$\lambda_1(s) \geq \text{Const.} \left( \frac{1}{\log(1/|s|)} \right)^2.$$

# Set up

$X$ : compact Kähler surface $S$ : compact Riemann surface
$f\colon X \to S$ : surjective proper holomorphic map with connected fibers
$\Delta \subset S$ : discriminant locus of $f\colon X \to S$
Set

$$S^o := S \setminus \Delta, \qquad X^o := X \setminus f^{-1}(\Delta), \qquad f^o := f|_{X^o}$$

Then $f^o\colon X^o \to S^o$ is a family of compact Riemann surfaces.

$\gamma$ : Kähler metric on $X$.
Set $X_s := f^{-1}(s)$, $\gamma_s = \gamma|_{X_s}$, $K_s$ the Gauss curvature of $(X_s, \gamma_s)$
Since $\min_{X_s} K_s \to -\infty$ as $s \to 0 \in \Delta$ and $K_s$ can be positive at some point, the Schoen-Wolpert-Yau theorem does not apply at once to the degeneration $f\colon X \to S$.

# Behavior of eigenvalues of the Laplacian

Let $\lambda_0(s) = 0 < \lambda_1(s) \le \lambda_2(s) \le \cdots$ be the eigenvalues of the Laplacian $\square_s = \bar\partial^* \bar\partial$ acting on $C^\infty(X_s)$ with respect to the induced metric $\gamma_s$.
For $s = 0$, regard $\square_0$ as the Friedrichs extension of the Laplacian acting on the smooth functions on $X_{0,\mathrm{reg}} = X_0 \setminus \mathrm{Sing}\, X_0$ with compact support.

## Fact (Y.)

*Assume that $X_0$ is reduced. Then, for each $k \in \mathbf{N}$, $\lambda_k(s)$ is a continuous function on $S$. Let*

$$N := \#\{ \text{irreducible components of } X_0\}$$

*Then*

$$\lim_{s\to 0} \lambda_k(s) = 0 \qquad (k \le N-1)$$

*and $\lambda_k(s)$, $k \ge N$ is uniformly bounded below by a positive constant.*

## Problem

Determine the asymptotic behavior of the small eigenvalues $\lambda_k(s)$ $(k \leq N-1)$ as $s \to 0$.

As for the first eigenvalue $\lambda_1(s)$, we have the following comparison from the mini-max principle. Let $\lambda_1^{\mathrm{hyp}}(s)$ be the first eigenvalue of $X_s$ with respect to the hyperbolic metric $\gamma_s^{\mathrm{hyp}}$ on $X_s$. To emphasis that $\gamma_s$ is induced from the metric $\gamma$ on $X$, write $\gamma_s^{\mathrm{ind}}$ for $\gamma_s = \gamma|_{X_s}$.

## Comparison of the first eigenvalues

$$\lambda_1(s) = \lambda_1^{\mathrm{ind}}(s) \geq m(s)\, \lambda_1^{\mathrm{hyp}}(s)$$

where

$$m(s) := \min_{X_s} \frac{\gamma_s^{\mathrm{hyp}}}{\gamma_s^{\mathrm{ind}}}.$$

# Case of degenerating family to a stable curve

When the family $f\colon X \to S$ is a degenerating family to a stable curve, due to Mazur's theorem, $\exists C > 0$ s.t. $m(s) \geq C > 0\ \forall\, s \in S^o = S \setminus \Delta$. Hence

$$\lambda_1(s) \geq \mathrm{Const.}\frac{1}{\log(1/|s|)}.$$

In this case, by making use of the mini-max principle, it is possible to prove

$$\lambda_{N-1}(s) \leq \mathrm{Const.}\frac{1}{\log(1/|s|)}.$$

Hence, for every degeneration to a stable curve, we get

**Theorem (Schoen-Wolpert-Yau + Mazur + mini-max principle)**

$$\frac{C_1}{\log(1/|s|)} \leq \lambda_1(s) \leq \cdots \leq \lambda_{N-1}(s) \leq \frac{C_2}{\log(1/|s|)} \qquad (s \to 0).$$

Now, let us consider the case where $X_0$ is NOT a stable curve.
By Cheeger's inequality, $\exists\,\alpha > 0$ such that

$$\lambda_1(s) \geq \mathrm{Const.}|s|^\alpha.$$

Since $X_0$ is not a stable curve, it can be possible that $\gamma_s^{\mathrm{hyp}}$ converges to $0$ on some components of $X_0$. There are examples such that $m(s)$ satisfies

$$\exists\,\alpha > 0 \qquad \text{s.t.} \qquad m(s) = \inf_{X_s}(\gamma_s^{\mathrm{hyp}}/\gamma_s^{\mathrm{ind}}) \leq C|s|^\alpha$$

Hence if $f\colon X \to S$ is NOT a degeneration to a stable curve, the estimate for $\lambda_1(s)$ obtained from the Schoen-Wolpert-Yau theorem and the mini-max principle seems to be of the above type: $\exists\,\alpha > 0$ such that

$$\lambda_1(s) \geq \mathrm{Const.}|s|^\alpha.$$

**Problem**

Is this estimate optimal?

# Main Results (preliminary)

To state the main theorem, let us introduce some notation.

Let $f\colon X \to S$ be a surjective holomorphic map from a compact complex surface $X$ to a compact Riemann surface $S$ with connected fibers.
The direct image sheaf $f_* K_{X/S}$ is a locally free sheaf on $S$. Here $K_{X/S} = K_X \otimes f^* K_S^{-1}$ is the relative canonical line bundle.
Let $\{\phi_1(s), \ldots, \phi_g(s)\}$ be a free basis of $f_* K_{X/S}$ near $0 \in \Delta$.

By the theory of variations of Hodge structures, the following is classical.

**Theorem (Griffiths, Schmid, Eriksson-Freixas-Mourougane)**

$\exists\ \alpha_1 \in \mathbf{Q}$, $\nu_1 \in \mathbf{Z}_{\geq 0}$, $\gamma_1 \in \mathbf{R}$ s.t. as $s \to 0$,

$$\log\det\left(\int_{X_s} \phi_i(s) \wedge \overline{\phi_j(s)}\right)_{1\leq i,j\leq g} = \alpha_1 \log|s|^2 + \nu_1 \log\log\frac{1}{|s|} + \gamma_1 + o(1).$$

Here $\alpha_1$ and $\nu_1$ are given explicitly in terms of the monodromy action on the limit mixed Hodge structure.

# Main Results (preliminary)

Let $H$ be an ample line bundle over $X$ endowed with a Hermitian metric $h_H$ with semi-positive first Chern form. We set $H_s = H|_{X_s}$.
$f_* K_{X/S}(H)$ is a locally free sheaf on $S$. Let $\{\psi_1(s), \ldots, \psi_{g+m}(s)\}$, $m = \deg H_s + g - 1$, be a free basis of $f_* K_{X/S}(H)$ around $0 \in \Delta$.
By the representability of the cohomology group $H^q(f^{-1}(U), K_X(H))$ by the harmonic forms due to Takegoshi, the existence of an asymptotic expansion of fiber integrals due to Barlet, and the non-degeneracy of the $L^2$-metric due to Mourougane-Takamaya, we still have a similar asymptotic expansion as $s \to 0$ in this case.

**Theorem**

$\exists\ \alpha_2 \in \mathbf{Q}$, $\nu_2 \in \mathbf{Z}_{\geq 0}$, and $\gamma_2 \in \mathbf{R}$ s.t. as $s \to 0$,

$$\log\det\left(\int_{X_s} h_H(\psi_i(s) \wedge \overline{\psi_j(s)})\right) = \alpha_2 \log|s|^2 + \nu_2 \log\log\frac{1}{|s|} + \gamma_2 + o(1).$$

# Main Results

Recall that $f\colon X \to S$ is a surjective proper holomorphic map from a compact complex surface $X$ to a compact Riemann surface $S$ with connected fibers.
$X_s$ is endowed with the Kähler metric $\gamma_s = \gamma|_{X_s}$.
$X_0$ has $N$ irreducible components: $X_0 = C_1 \cup \cdots \cup C_N$.
(We do NOT assume that $X_0$ is a stable curve.)
Recall that $\nu_1, \nu_2 \in \mathbf{Z}_{\geq 0}$ are the coefficients of the subleading term of the determinant of the period integrals. We set

$$\nu = \nu_2 - \nu_1 \in \mathbf{Z}.$$

## Theorem (Dai-Y.)

*If $N \geq 2$ and $X_0$ is reduced, then $\nu > 0$ and $\exists\, c \in \mathbf{R}_{>0}$ such that*

$$\prod_{j=1}^{N-1} \lambda_j(s) = \frac{c + o(1)}{(\log(1/|s|))^{\nu}} \qquad (s \to 0).$$

Since $\lambda_i(s) \le 1$ for $|s| \ll 1$ and $i < N$, we have

$$\lambda_i(s) \ge \prod_{j=1}^{N-1} \lambda_j(s) = \frac{c + o(1)}{(\log(1/|s|))^{\nu}}.$$

Since

$$\lambda_1(s)^{N-1} \le \prod_{j=1}^{N-1} \lambda_j(s) = \frac{c + o(1)}{(\log(1/|s|))^{\nu}} \le \lambda_{N-1}(s)^{N-1},$$

we get

$$\lambda_1(s) \le \frac{c^{1/(N-1)} + o(1)}{(\log(1/|s|))^{\nu/(N-1)}} \le \lambda_{N-1}(s).$$

## Theorem (Dai-Y.)

$\exists$ *constants* $C_1, C_2 > 0$ *such that for all* $s \in S \setminus \Delta$,

$$\frac{C_1}{(\log(1/|s|))^{\nu}} \le \lambda_1(s) \le \frac{C_2}{(\log(1/|s|))^{\nu/(N-1)}}.$$

# Estimate for the samll eigenvalues

It is possible to construct an orthonormal system of functions $\{\varphi_1(s), \ldots, \varphi_{N-1}(s)\} \subset C^\infty(X_s)$ such that as $s \to 0$,

$$(\varphi_i(s), 1)_{L^2} = o(1), \quad \|\varphi_i(s)\|_{L^2} = 1 + o(1), \quad \|d\varphi_i(s)\|_{L^2}^2 \leq \frac{C}{\log(1/|s|)}.$$

Hence, by the mini-max principle, we have the following:

## Theorem (Dai-Y.)

$\exists$ constants $C_1(i), C_2(i) > 0$ such that for all $s \in S^o$,

$$\frac{C_1(i)}{(\log(1/|s|))^\nu} \leq \lambda_i(s) \leq \frac{C_2(i)}{\log(1/|s|)} \qquad (1 \leq i \leq N-1).$$

For $i = N-1$, one has

$$\frac{C_3}{(\log(1/|s|))^{\nu/(N-1)}} \leq \lambda_{N-1}(s) \leq \frac{C_2(N-1)}{\log(1/|s|)}.$$

# A Conjecture

Recall that

$$\prod_{i=1}^{N-1} \lambda_i(s) = \frac{c + o(1)}{(\log(1/|s|))^{\nu}} \qquad (s \to 0).$$

## Conjecture

$$\nu = \nu_2 - \nu_1 = N - 1 = \#\{\text{irreducible components of } X_0\} - 1.$$

In particular,

$$(*) \qquad \prod_{i=1}^{N-1} \lambda_i(s) = \frac{c + o(1)}{(\log(1/|s|))^{N-1}} \qquad (s \to 0).$$

# A consequence of the conjecture

Since

$$\frac{C_3}{(\log(1/|s|))^{\nu/(N-1)}} \leq \lambda_{N-1}(s) \leq \frac{C_2(N-1)}{\log(1/|s|)},$$

we get

$$\lambda_{N-1}(s) \sim \frac{1}{\log(1/|s|)}.$$

Then, by $(*)$,

$$\lambda_{N-2}(s)^{N-2} \geq \prod_{i=1}^{N-2} \lambda_i(s) \sim \frac{1}{(\log(1/|s|))^{N-2}}.$$

Namely,

$$\lambda_{N-2}(s) \geq \frac{\text{Const.}}{\log(1/|s|)}$$

# A consequence of the conjecture

On the other hand, by the mini-max principle,

$$\lambda_{N-2}(s) \leq \frac{C_2(N-2)}{\log(1/|s|)}.$$

Hence

$$\lambda_{N-2}(s) \sim \frac{1}{\log(1/|s|)}.$$

Inductively, we have

$$\lambda_i(s) \sim \frac{1}{\log(1/|s|)} \qquad (1 \leq i \leq N-1).$$

This is an analogue of the Schoen-Wolpert-Yau theorem for the degenerations of Riemann surfaces to a stable curve.

# Outline of the proof of the Main Theorem

Recall that $H$ is an ample line bundle endowed with a Hermitian metric $h_H$ with semi-positive curvature. Set

$$(L, h) = (H^{-1}, h_H^{-1})$$

To prove the theorem, we compare the analytic torsions $\tau(X_s, \mathcal{O}_{X_s})$ and $\tau(X_s, L_s)$, where $L_s = L|_{X_s}$.

Let us recall the notion of analytic torsion.

# The spectral zeta function

Let $(M, h_M)$ be a compact Riemann surface with a Kähler metric.
Let $(E, h_E)$ be a holomorphic Hermitian vector bundle over $M$.
Let $\Box_{0,q} = (\bar{\partial} + \bar{\partial}^*)^2$ be the Laplacian acting on $A_M^{0,q}(E)$.
Let $\zeta_{0,q}(s)$ be the zeta function of $\Box_{0,q}$

$$
\begin{aligned}
\zeta_{0,q}(s) &= \sum_{\lambda \in \sigma(\Box_{0,q}) \setminus \{0\}} \lambda^{-s} \dim E(\lambda, \Box_{0,q}) \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \{\operatorname{Tr} e^{-t\Box_{0,q}} - h^{0,q}(E)\} t^{s-1} \, dt
\end{aligned}
$$

where $E(\lambda, \Box_{0,q})$ is the eigenspace of $\Box_{0,q}$ with eigenvalue $\lambda$.

**Fact**

*$\zeta_{0,q}(s)$ converges absolutely when $\Re s > \dim M$, admits a meromorphic continuation to $\mathbb{C}$, and is holomorphic at $s = 0$.*

# Analytic torsion

## Definition (Ray-Singer, Bismut-Gillet-Soulé)

The analytic torsion of $(M, E)$ with respect to the metrics $h_M$, $h_E$ is the real number

$$\tau(M,E) = \exp[-\sum_{q\geq 0}(-1)^q q\,\zeta'_{0,q}(0)] = e^{\zeta'_{0,1}(0)} = e^{\zeta'_{0,0}(0)}.$$

We have the following formal identity:

$$\tau(M,E) = \left(\prod_{\lambda\in\sigma(\Box_{0,0})\setminus\{0\}} \lambda^{\dim E(\lambda,\Box_{0,0})}\right)^{-1}$$

Of course, the right hand side does not converge.

# Outline of the proof 1: Partial analytic torsion

Recall that $L$ is a holomorphic line bundle on $X$. Set $L_s := L|_{X_s}$. Let

$$K^{L_s}(t,x,x) \sim \frac{a_0(x,L_s)}{t} + a_1(x,L_s) + O(t) \qquad (t \to 0)$$

be the asymptotic expansion of the kernel of $e^{-t\square_{L_s}}$, where $\square_{L_s} = \bar{\partial}^*_{L_s}\bar{\partial}_{L_s}$. Then

$$\begin{aligned}\log\tau(X_s,L_s) &= \int_0^1 \frac{dt}{t}\int_{X_s}\{K^{L_s}(t,x,x) - \frac{a_0(x,L_s)}{t} - a_1(x,L_s)\}dv_x \\ &+ \int_1^\infty \frac{dt}{t}\{\int_{X_s} K^{L_s}(t,x,x)dv_x - h^{0,q}(L_s)\} \\ &- \Gamma'(1)\{\int_{X_s} a_0(x,L_s)dv_x - h^0(L_s)\}\end{aligned}$$

where $\int_{X_s} a_0(x,L_s)dv_x$ is a topological constant independent of $s \in S$.

Let $\Omega$ be an open neighborhood of the singular point $o \in \operatorname{Sing} X_0$. Define the partial analytic torsions of $(X_s, L_s)$ by

$$\log \tau^{\Omega}_{[0,1]}(X_s, L_s) = \int_0^1 \frac{dt}{t} \int_{\Omega \cap X_s} \{K^{L_s}(t,x,x) - \frac{a_0(x, L_s)}{t} - a_1(x, L_s)\} dv_x$$

$$\log \tau^{X_s \setminus \Omega}_{[0,1]}(X_s, L_s) = \int_0^1 \frac{dt}{t} \int_{X_s \setminus \Omega} \{K^{L_s}(t,x,x) - \frac{a_0(x, L_s)}{t} - a_1(x, L_s)\} dv_x$$

$$\log \tau_{[1,\infty]}(X_s, L_s) = \int_1^\infty \frac{dt}{t} \{ \int_{X_s} K^{L_s}(t,x,x) dv_x - h^0(L_s)\}$$

Then $\exists$ constant $C$ independent of $s \in S^o$ such that

$$\log \tau(X_s, L_s) = \log \tau^{\Omega}_{[0,1]}(X_s, L_s) + \log \tau^{X_s \setminus \Omega}_{[0,1]}(X_s, L_s) + \log \tau_{[1,\infty]}(X_s, L_s) + C$$

# Estimate for the partial analytic torsion I

Recall that $L^{-1}$ is an ample line bundle on $X$. Assume that $(L,h)|_\Omega$ *is a* *trivial holomorphic Hermitian line bundle on* $\Omega \supset \mathrm{Sing}\, X_0$ and that $c_1(L,h) \leq 0$ over $X$. (Such a metric $h$ exists because $\mathrm{Sing}\, X_0$ is isolated.) In what follows, we adopt the following definition:

$$O(1) = const. + o(1)$$

## Lemma

$$\log \tau^{\Omega}_{[0,1]}(X_s, \mathcal{O}_{X_s}) - \log \tau^{\Omega}_{[0,1]}(X_s, L_s) = O(1) \qquad (s \to 0).$$

## Lemma

$$\log \tau^{X_s \setminus \Omega}_{[0,1]}(X_s, \mathcal{O}_{X_s}) = O(1), \qquad \log \tau^{X_s \setminus \Omega}_{[0,1]}(X_s, L) = O(1) \qquad (s \to 0).$$

# Estimate for the partial analytic torsion II

Let $k(t,x,y)$ be the heat kernel of the Laplacain $\square_s$ acting on $C^\infty(X_s)$. Since $L^{-1}$ is ample, we have $H^0(X_s, L_s) = 0$. Recall

$$\log \tau_{[1,\infty]}(X_s, \mathcal{O}_{X_s}) = \int_1^\infty \frac{dt}{t}\left(\int_{X_s} k(t,x,x)dv_x - 1\right) = \int_1^\infty \sum_{\lambda_i(s)>0} e^{-t\lambda_i(s)} \frac{dt}{t}$$

$$\log \tau_{[1,\infty]}(X_s, L_s) = \int_1^\infty \frac{dt}{t} \int_{X_s} K^{L_s}(t,x,x)dv_x = \int_1^\infty \sum_{\mu_i(s)>0} e^{-t\mu_i(s)} \frac{dt}{t}$$

**Lemma**

$$\log \tau_{[1,\infty]}(X_s, \mathcal{O}_{X_s}) = \log \left\{ \prod_{i=1}^{N-1} \lambda_i(s) \right\} + O(1) \qquad (s \to 0).$$

**Lemma**

$$\log \tau_{[1,\infty]}(X_s, L_s) = O(1) \qquad (s \to 0).$$

# Asymptotic behavior of analytic torsion I

To state some results concerning the asymptotic behavior of the analytic torsion, let us recall the notion of Milnor number.

If $p \in \operatorname{Sing} X_0$ and $f(x,y) \in \mathbb{C}\{x,y\}$ is a local defining equation of $(X_0, p)$, then

$$\mu(X_0, p) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f_x(x,y), f_y(x,y))}$$

is the Milnor number of the singularity $(X_0, p)$. The total Milnor number of the singular fiber $X_0$ is defined as:

$$\mu(\operatorname{Sing} X_0) = \sum_{p \in \operatorname{Sing} X_0} \mu(X_0, p).$$

# Asymptotic behavior of analytic torsion II

Applying the Bismut-Lebeau embedding theorem for Quillen metrics to the family of embeddings $X_s = f^{-1}(s) \hookrightarrow X$, we have the following theorems.

**Theorem (Y.)**

$$\log \tau(X_s, \mathcal{O}_{X_s}) + \log \det \left( \int_{X_s} \phi_i(s) \wedge \overline{\phi_j(s)} \right)_{1 \le i,j \le g}$$
$$= -\frac{1}{6}\mu(\mathrm{Sing}\, X_0)\ \log |s|^2 + O(1) \qquad (s \to 0).$$

**Theorem (Y.)**

$$\log \tau(X_s, L_s) + \log \det \left( \int_{X_s} h_H(\psi_i(s) \wedge \overline{\psi_j(s)}) \right)_{1 \le i,j \le g+m}$$
$$= -\frac{1}{6}\mu(\mathrm{Sing}\, X_0)\ \log |s|^2 + O(1) \qquad (s \to 0).$$

# Asymptotic behavior of analytic torsion III

## Theorem (Y.)

*One has the following asymptotic expansions as* $s \to 0$*:*

$$\log \tau(X_s, \mathcal{O}_{X_s}) = \kappa_1 \log |s|^2 - \nu_1 \log(\log \frac{1}{|s|^2}) + O(1) \qquad (s \to 0),$$

$$\log \tau(X_s, L) = \kappa_2 \log |s|^2 - \nu_2 \log(\log \frac{1}{|s|^2}) + O(1) \qquad (s \to 0).$$

*Here,* $\kappa_1, \kappa_2 \in \mathbb{Q}$*. Moreover, the leading terms coincide:*

$$\kappa_1 = \kappa_2.$$

*In particular, as* $s \to 0$*,*

$$\log \tau(X_s, \mathcal{O}_{X_s}) - \log \tau(X_s, L) = (\nu_2 - \nu_1) \log(\log \frac{1}{|s|^2}) + O(1).$$

# Asymptotic behavior of analytic torsion IV

To prove this theorem, we consider the semistable reduction of $f\colon X \to S$.

## Theorem (Mumford et al: Semistable reduction theorem '73)

*Let $(T, 0)$ be another pointed unit disc of $\mathbb{C}$. Then there exist $m \in \mathbb{Z}$ and a family $\pi\colon (Y, Y_0) \to (T, 0)$ with the following properties.*

- *$\pi\colon Y \setminus Y_0 \to T \setminus \{0\}$ is the pull-back of $f\colon X \setminus X_0 \to S \setminus \{0\}$ by the map $\mu\colon T \to S$, $\mu(t) = t^m$.*
- *$Y$ is smooth and $Y_0$ is a reduced, normal crossing divisor, i.e., locally,*

$$Y_t = \{z_0 z_1 = t\}.$$

- *There is a map $F\colon Y \to X$ sending the fibers of $\pi$ to fibers of $f$ such that*

$$f \circ F = \mu \circ f.$$

# Asymptotic behavior of analytic torsion IV

This situation can be summarized as the commutative diagram:

$$\begin{array}{ccc} F\colon (Y,Y_0) & \to & (X,X_0) \\ \pi\downarrow & & \downarrow f \\ \mu\colon (T,0) & \to & (S,0) \end{array}$$

with $s=t^m$, and $Y_0$ reduced and normal crossing divisor, and $Y$ smooth.

## Why $\kappa_1=\kappa_2$?

The coefficients $\kappa_1$, $\kappa_2$ of the leading term of the asymptotic expansion of the analytic torsion can be expressed as the integral of certain characteristic classes determined by the pair $(U,\mathrm{Sing}\,X_0)$, the bundle $\mathcal{O}_X|_U$, $H|_U$ and the pair $(F^{-1}(U),F^{-1}(\mathrm{Sing}\,X_0))$. Here $U$ is an arbitrary small neighborhood of $\mathrm{Sing}\,X_0$ in $X$. Hence, if $H|_U\cong\mathcal{O}_X|_U$, then $\kappa_1=\kappa_2$. (This explains why we must assume that $f$ has only isolated critical points on $X_0$.)

# Proof of the Main Theorem I

We have

$$\log\tau(X_s,\mathcal{O}_{X_s}) = \log\tau^{\Omega}_{[0,1]}(X_s,\mathcal{O}_{X_s}) + \log\tau^{X_s\setminus\Omega}_{[0,1]}(X_s,\mathcal{O}_{X_s}) + \log\tau_{[1,\infty]}(X_s,\mathcal{O}_{X_s}) + C_1,$$

$$\log\tau(X_s,L_s) = \log\tau^{\Omega}_{[0,1]}(X_s,L_s) + \log\tau^{X_s\setminus\Omega}_{[0,1]}(X_s,L_s) + \log\tau_{[1,\infty]}(X_s,L_s) + C_2,$$

where $C_1$ and $C_2$ are topological terms.

# Proof of the Main Theorem II

Since

$$\log \tau^{\Omega}_{[0,1]}(X_s, \mathcal{O}_{X_s}) - \log \tau^{\Omega}_{[0,1]}(X_s, L_s) = O(1),$$

$$\log \tau^{X_s \setminus \Omega}_{[0,1]}(X_s, \mathcal{O}_{X_s}) - \log \tau^{X_s \setminus \Omega}_{[0,1]}(X_s, L_s) = O(1),$$

$$\log \tau_{[1,\infty]}(X_s, \mathcal{O}_{X_s}) - \log \tau_{[1,\infty]}(X_s, L_s) = \log\{\prod_{i=1}^{N-1} \lambda_i(s)\} + O(1),$$

we get

$$\log \tau(X_s, \mathcal{O}_{X_s}) - \log \tau(X_s, L_s) = \log\{\prod_{i=1}^{N-1} \lambda_i(s)\} + O(1).$$

Namely, the product of the small eigenvalues appears as the difference of the analytic torsions.

# Proof of the Main Theorem III

On the other hand, we have

$$\log \tau(X_s, \mathcal{O}_{X_s}) - \log \tau(X_s, L) = (\nu_2 - \nu_1) \log(\log \frac{1}{|s|^2}) + O(1) \qquad (s \to 0).$$

Comparing this with the previous formula

$$\log \tau(X_s, \mathcal{O}_{X_s}) - \log \tau(X_s, L_s) = \log\{\prod_{i=1}^{N-1} \lambda_i(s)\} + O(1),$$

we get the desired formula:

$$\log\{\prod_{i=1}^{N-1} \lambda_i(s)\} = (\nu_2 - \nu_1) \log(\log \frac{1}{|s|^2}) + O(1) \qquad (s \to 0). \quad \square$$

# A conjecture in higher dimensions

Let $f\colon X \to S$ be a one-parameter degenerating family of compact Kähler manifolds of dimension $n$ such that $f$ has only isolated critical points. Let $\{0 = \cdots = 0 < \lambda_1(s) \leq \lambda_2(s) \leq \cdots \leq \lambda_k(s) \leq \cdots\}$ be the eigenvalues of the Laplacian $\Box_{n,0}$ acting on $(n,0)$-forms on $X_s = f^{-1}(s)$ with respect to the metric induced from the Kähler metric on $X$.
Then the following hold:

- For all $k \in \mathbb{N}$, $\lambda_k(s)$ extends to a continuous function on $S$.
- Let $\lambda_1(s) \leq \cdots \leq \lambda_N(s)$ be the small eigenvalues of $\Box_{n,0}$. Then there exists $\nu \in \mathbb{N}$, $c \in \mathbb{R}_{>0}$ such that

$$\prod_{k=1}^{N} \lambda_k(s) = \frac{c + o(1)}{(\log(1/|s|))^{\nu}} \qquad (s \to 0).$$

- In fact, for $1 \leq k \leq N$, one has $\lambda_k(s) \sim 1/\log(1/|s|)$ as $s \to 0$.