# Expected centre of mass of the random Kodaira embedding

Yoshinori Hashimoto

Tokyo Institute of Technology

4th November 2021
The 27th Symposium on Complex Geometry

# Kodaira embedding and the centre of mass

# Kodaira embedding

Let $X \subset \mathbb{CP}^{N-1}$ be a complex smooth projective variety, i.e. $X$ is a complex manifold defined as an algebraic set given by the vanishing of various homogeneous polynomials in $N$ variables.

**Theorem (Kodaira 1954)**

*If $X$ is a complex manifold with an ample line bundle $L$ (i.e. $L$ is "positive" in a certain sense), there exists a holomorphic embedding*

$$\iota : X \hookrightarrow \mathbb{P}(H^0(X, L^{\otimes k})^\vee) \cong \mathbb{CP}^{N-1}$$

*for all $k \gg 0$, where $N := \dim_{\mathbb{C}} H^0(X, L^{\otimes k})$.*

$\iota(X) \subset \mathbb{CP}^{N-1}$ is an algebraic variety by Chow's theorem.

# Centre of mass

## Definition

The **centre of mass** of the Kodaira embedding $\iota : X \hookrightarrow \mathbb{CP}^{N-1}$ is an $N \times N$ hermitian matrix defined by

$$\bar{\mu}_X(\iota) := \int_{\iota(X)} \frac{Z_i \bar{Z}_j}{\sum_{m=1}^{N} |Z_m|^2} \omega_{FS}^n,$$

where $[Z_1 : \cdots : Z_N]$ is the homogeneous coordinates for $\mathbb{CP}^{N-1}$, $\omega_{FS}$ is the Fubini–Study metric on $\mathbb{CP}^{N-1}$, and $n = \dim_{\mathbb{C}} X$.

Equivalently, $\bar{\mu}_X(\iota)$ is the integral of the moment map

$$\mu : \mathbb{CP}^{N-1} \to \sqrt{-1}\mathfrak{u}(N)^{\vee} \cong \sqrt{-1}\mathfrak{u}(N)$$

for $U(N) \curvearrowright \mathbb{CP}^{N-1}$ over $\iota(X)$, i.e. $\bar{\mu}_X(\iota) = \int_{\iota(X)} \mu \, \omega_{FS}^n$.

# Centre of mass of the displaced embedding

Important point: natural linear action $GL(N, \mathbb{C}) \curvearrowright \mathbb{CP}^{N-1}$ induces an action $\iota \mapsto g \cdot \iota$ on the Kodaira embedding $\iota : X \hookrightarrow \mathbb{CP}^{N-1}$.

**Theorem (Zhang 1996, Luo 1998)**

*There exists $g \in GL(N, \mathbb{C})$ such that $\bar{\mu}_X(g \cdot \iota)$ is a constant multiple of the identity matrix if and only if the embedding $\iota : X \hookrightarrow \mathbb{CP}^{N-1}$ is Chow stable.*

1. **Chow stability** is a classical stability condition in Geometric Invariant Theory which is important in constructing a moduli space of varieties embedded in $\mathbb{CP}^{N-1}$ (Chow scheme).
2. Whether a given (embedded) variety $\iota : X \hookrightarrow \mathbb{CP}^{N-1}$ is Chow stable or not is a very difficult problem in general.

# Example: trivial embedding of $\mathbb{CP}^{N-1}$

**Lemma**

*Let $\iota : \mathbb{CP}^{N-1} \hookrightarrow \mathbb{CP}^{N-1}$ be the identity map. Then for any $g \in GL(N, \mathbb{C})$, $\bar{\mu}_{\mathbb{CP}^{N-1}}(g \cdot \iota)$ is a constant multiple of the identity matrix.*

Set $[Z'_1 : \cdots : Z'_N] = g \cdot [Z_1 : \cdots : Z_N]$. On an open dense subset $\{Z'_1 \neq 0\}$ of $\mathbb{CP}^{N-1}$, use polar coordinates $Z'_i/Z'_1 = r_i e^{\sqrt{-1}\theta_i}$, $i \geq 2$, to find

$$\bar{\mu}_{\mathbb{CP}^{N-1}}(g \cdot \iota) = \int_{\mathbb{CP}^{N-1}} \frac{Z'_i \bar{Z'}_j}{\sum_{m=1}^N |Z'_m|^2} \omega_{FS'}^{N-1}$$

$$= \int_{\mathbb{R}^{N-1}_{>0}} \mathrm{Jac}(\vec{r}) d\vec{r} \int_{[0,2\pi]^{N-1}} \frac{r_i r_j e^{\sqrt{-1}(\theta_i - \theta_j)}}{1 + \sum_{m=2}^N r_m^2} d\vec{\theta}.$$

Off-diagonal entries are zero by the periodicity of the angle variables. Diagonal entries are invariant under permutation.
$\Rightarrow \bar{\mu}_{\mathbb{CP}^{N-1}}(g \cdot \iota) = \mathrm{const.id}_N$.

# Example: Veronese embedding

## Lemma

*Let $\iota : \mathbb{CP}^{m-1} \hookrightarrow \mathbb{CP}^{N-1}$ be the degree $d$ Veronese embedding*

$$[z_1 : \cdots : z_m] \mapsto [Z_1 : \cdots : Z_N] := [z_1^d : z_1^{d-1} z_2 : \cdots : z_m^d].$$

*Then, $\bar{\mu}_{\mathbb{CP}^{m-1}}(g \cdot \iota)$ is a constant multiple of the identity for some diagonal matrix $g \in GL(N, \mathbb{C})$.*

Exactly as in the previous example, we find that the off-diagonal entries of $\bar{\mu}_{\mathbb{CP}^{m-1}}(g \cdot \iota)$ must be zero by the computation in polar coordinates. Unlike the previous one, the diagonal entries are not invariant under permutation, but we may scale each $Z_i$ by an appropriate diagonal matrix $g$ to get $\bar{\mu}_{\mathbb{CP}^{m-1}}(g \cdot \iota) = \text{const.}\mathrm{id}_N$.

# Centre of mass and canonical Kähler metrics

- The embedding $g \cdot \iota : X \hookrightarrow \mathbb{CP}^{N-1}$ is called **balanced** if $\bar{\mu}_X(g \cdot \iota)$ is a constant multiple of the identity matrix $\iff g \cdot \iota$ attains a "zero of the moment map".
- The restriction of the Fubini–Study metric on $\mathbb{CP}^{N-1}$ by the balanced embedding is called the **balanced metric**.
- Theorem of Donaldson (2001): a constant scalar curvature Kähler metric can be approximated by a sequence of balanced metrics (which implies asymptotic Chow stability), when $X$ has discrete automorphisms.
- Generalisations to the non-discrete automorphisms case done by Mabuchi, Sano–Tipler, Seyyedali, H.

Upshot: $\bar{\mu}_X(g \cdot \iota)$ is an interesting quantity that depends on $g$ and $\iota$ in a complicated nonlinear manner and captures subtle geometric properties of $\iota : X \hookrightarrow \mathbb{CP}^{N-1}$.

# Random Kodaira embedding

Suppose that we displace the Kodaira embedding $\iota : X \hookrightarrow \mathbb{CP}^{N-1}$ by a *random* element $g \in GL(N, \mathbb{C})$.

## Problem

*Let $d\sigma$ be a probability measure on $GL(N, \mathbb{C})$. What is the expectation of the centre of mass*

$$\mathbb{E}[\bar{\mu}_X(g \cdot \iota)] = \int_{GL(N,\mathbb{C})} \bar{\mu}_X(g \cdot \iota) d\sigma$$

*with respect to $d\sigma$?*

In spite of its apparent simplicity, this is a nontrivial problem because $\bar{\mu}_X(g \cdot \iota)$ depends nonlinearly on $g$.

Remark: random matrices and Kähler metrics also considered by Ferrari–Klevtsov–Zelditch.

# Results

# Main result

## Theorem (H. 2020)

*Let $d\sigma$ be a probability measure on $GL(N, \mathbb{C})$ induced by the fibration $U(N) \to GL(N, \mathbb{C}) \twoheadrightarrow GL(N, \mathbb{C})/U(N)$ with*

- *the Haar measure on the fibre $U(N)$,*
- *an absolutely continuous unitarily invariant measure $d\sigma_B$ of finite volume on the base $GL(N, \mathbb{C})/U(N)$.*

*Then, for any $\iota : X \hookrightarrow \mathbb{CP}^{N-1}$, the expected centre of mass is a constant multiple of the identity matrix:* $\mathbb{E}[\bar{\mu}_X(g \cdot \iota)] = \text{const.}\mathrm{id}_N$.

## Example (of the measure on the base)

The measure on $GL(N, \mathbb{C})/U(N)$, the set of all positive definite hermitian matrices, may be given by the **Gaussian unitary ensemble** $d\sigma_B = \exp(-\mathrm{tr}(H^2))dH$, where $dH$ is the Lebesgue measure on the space of hermitian matrices.

# Unitary version of the main result

**Theorem (H. 2020)**

*Let $d\sigma_U$ be the Haar measure on the unitary group $U(N)$. For any smooth projective variety $\iota : X \hookrightarrow \mathbb{CP}^{N-1}$, the expectation of the unitarily displaced centre of mass*

$$\mathbb{E}_U[\bar{\mu}_X(u \cdot \iota)] := \int_{U(N)} \bar{\mu}_X(u \cdot \iota) d\sigma_U$$

*is a constant multiple of the identity matrix.*

The proof of this theorem is much easier than the previous one, since the action by $u \in U(N)$ simplifies as

$$\bar{\mu}_X(u \cdot \iota) = \int_{\iota(X)} \frac{u \cdot Z_i \overline{u \cdot Z_j}}{\sum_{m=1}^N |u \cdot Z_m|^2} (u^* \omega_{FS}^n) = u \left( \int_{\iota(X)} \frac{Z_i \bar{Z}_j}{\sum_{m=1}^N |Z_m|^2} \omega_{FS}^n \right) u^{-1},$$

noting that $U(N)$ acts isometrically.

# Strategy of the proof: incidence variety

We need to compute

$$\mathbb{E}[\bar{\mu}_X(g \cdot \iota)] = \int_{GL(N,\mathbb{C})} \bar{\mu}_X(g \cdot \iota) d\sigma$$

$$= \int_{g \in GL(N,\mathbb{C})} d\sigma \int_{x' \in g \cdot \iota(X)} \frac{Z_i(x')\overline{Z_j(x')}}{\sum_{m=1}^N |Z_m(x')|^2} \omega_{FS}^n(x').$$

We do so by "exchanging" the order of integrals over $\iota(X)$ and $GL(N,\mathbb{C})$; we first define the incidence variety

$$\mathcal{I} := \{(g, g \cdot x) \in GL(N,\mathbb{C}) \times \mathbb{CP}^{N-1} \mid x \in \iota(X)\},$$

with $\pi : \mathcal{I} \twoheadrightarrow \iota(X)$, $\pi(g, g \cdot x) := g^{-1}(g \cdot x) = x$, and an appropriate measure $d\tau_{\mathcal{I}}$ on $\mathcal{I}$. so that we re-formulate the above integral as

$$\mathbb{E}[\bar{\mu}_X(g \cdot \iota)] = \int_{\mathcal{I}} \frac{g \cdot Z_i(x)\overline{g \cdot Z_j(x)}}{\sum_{m=1}^N |g \cdot Z_m(x)|^2} \frac{g^* \omega_{FS}^n(x)}{\omega_{FS}^n(x)} d\tau_{\mathcal{I}}.$$

# Coarea formula

**Theorem (Coarea formula, Federer)**

*Let $(M, g_M)$ and $(S, g_S)$ be smooth Riemannian manifolds with $\dim M > \dim S$. Suppose that $f : M \to S$ is a smooth surjection. Then for any measurable function $f$ on $M$ we have*

$$\int_{p \in M} \phi(p) d\sigma_{g_M} = \int_{q \in S} d\sigma_{g_S} \int_{p \in M_{\mathrm{reg}} \cap f^{-1}(q)} \frac{\phi(p)}{Jf(p)} d\sigma_{g_M}|_{f^{-1}(q)},$$

*where $M_{\mathrm{reg}}$ is the regular locus of $f$ and $Jf$ is the generalised Jacobian of $f$.*

Example: if $M = S \times S'$ is a Riemannian product, we have $Jf \equiv 1$ and the above theorem reduces to the classical Fubini theorem.

This theorem can be generalised to non-smooth $f$ (Federer).

# Strategy of the proof: application of the coarea formula

We apply the coarea formula to $\pi : \mathcal{I} \to \iota(X)$, $\pi(g, g \cdot x) := x$, to find that $\mathbb{E}[\bar{\mu}_X(g \cdot \iota)]$ equals

$$\int_{x \in \iota(X)} \omega_{FS}^n(x) \int_{g \in GL(N,\mathbb{C})} \frac{g \cdot Z_i(x) \overline{g \cdot Z_j(x)}}{\sum_{m=1}^N |g \cdot Z_m(x)|^2} \frac{1}{J\pi(g,x)} \frac{g^* \omega_{FS}^n(x)}{\omega_{FS}^n(x)} d\sigma,$$

where $J\pi$ is the generalised Jacobian of $\pi$ (I learned this trick from Jean-Yves Welschinger).

The key result is the following.

**Proposition**

*For any $g \in GL(N, \mathbb{C})$ and $x \in \iota(X)$ we have*

$$\frac{1}{J\pi(g,x)} \frac{g^* \omega_{FS}^n(x)}{\omega_{FS}^n(x)} = 1.$$

# Strategy of the proof: global Cartan decomposition

We are thus reduced to compute the following integral:

$$\mathbb{E}[\bar{\mu}_X(g\cdot\iota)] = \int_{x\in\iota(X)} \omega_{FS}^n(x) \int_{g\in GL(N,\mathbb{C})} \frac{g\cdot Z_i(x)\overline{g\cdot Z_j(x)}}{\sum_{m=1}^N |g\cdot Z_m(x)|^2} d\sigma.$$

We write $g = \eta\Lambda u$ ($\eta, u \in U(N)$ and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$), by recalling the global Cartan decomposition for $GL(N,\mathbb{C})$. Recalling our hypothesis that $d\sigma_B$ is unitarily invariant and absolutely continuous, we may write

$$d\sigma(g) = \rho(\Lambda) \prod_{1\leq i\neq j\leq N} |\lambda_i - \lambda_j|^2 d\Lambda\, d\sigma_U(\eta)\, d\sigma_U(u),$$

for some density function $\rho$ that depends only on $\Lambda$.
Example: $\rho(\Lambda) = \exp(-\sum_{i=1}^N \lambda_i^2)$ for the Gaussian unitary ensemble.
In what follows, we write $\rho'(\Lambda)$ for $\rho(\Lambda)\prod_{1\leq i\neq j\leq N}|\lambda_i - \lambda_j|^2$.

# Strategy of the proof: reduction to $\mathbb{CP}^{N-1}$

Thus, noting $g = \eta \Lambda u$, the integral over $GL(N, \mathbb{C})$ reduces to

$$\int_{g \in GL(N,\mathbb{C})} \frac{g \cdot Z_i(x) \overline{g \cdot Z_j(x)}}{\sum_{m=1}^N |g \cdot Z_m(x)|^2} d\sigma$$

$$= \int_{\mathbb{R}^N} \rho'(\Lambda) d\Lambda \int_{U(N)} d\sigma_U(\eta) \int_{U(N)} \frac{(\eta \Lambda u) \cdot Z_i(x) \overline{(\eta \Lambda u) \cdot Z_j(x)}}{\sum_{m=1}^N |(\eta \Lambda u) \cdot Z_m(x)|^2} d\sigma_U(u)$$

$$= \int_{\mathbb{R}^N} \rho'(\Lambda) d\Lambda \int_{U(N)} \eta \left( \int_{U(N)} \frac{(\Lambda u) \cdot Z_i(x) \overline{(\Lambda u) \cdot Z_j(x)}}{\sum_{m=1}^N |(\Lambda u) \cdot Z_m(x)|^2} d\sigma_U(u) \right) \eta^{-1} d\sigma_U(\eta)$$

For each $p \in \mathbb{CP}^{N-1}$ we have $\mathrm{Stab}_{U(N)}(p) = U(1) \times U(N-1)$, which implies that the above integral in $u$ is over $U(N)/U(1) \times U(N-1) \cong \mathbb{CP}^{N-1}$. By the uniqueness of the group invariant measure on homogeneous spaces, we find that $d\sigma_U(u)$ must be a constant multiple of $\omega_{FS}^{N-1}$.

# Strategy of the proof: computation over $\mathbb{CP}^{N-1}$

We are thus reduced to the computation over $\mathbb{CP}^{N-1}$, where explicit calculations in polar coordinates are available. Exactly as we saw in the previous example, we find the following.

**Proposition**

$\int_{U(N)} \frac{(\Lambda u)\cdot Z_i(x)\overline{(\Lambda u)\cdot Z_j(x)}}{\sum_{m=1}^N |(\Lambda u)\cdot Z_m(x)|^2} d\sigma_U(u)$ *is a diagonal matrix.*

Thus, the integral

$$\int_{g\in GL(N,\mathbb{C})} \frac{g\cdot Z_i(x)\overline{g\cdot Z_j(x)}}{\sum_{m=1}^N |g\cdot Z_m(x)|^2} d\sigma$$

$$= \int_{\mathbb{R}^N} \rho'(\Lambda)d\Lambda \int_{U(N)} \eta \left( \int_{U(N)} \frac{(\Lambda u)\cdot Z_i(x)\overline{(\Lambda u)\cdot Z_j(x)}}{\sum_{m=1}^N |(\Lambda u)\cdot Z_m(x)|^2} d\sigma_U(u) \right) \eta^{-1} d\sigma_U(\eta)$$

over $GL(N,\mathbb{C})$ becomes a constant multiple of the identity matrix.

# Strategy of the proof: conclusion

Thus, for each $x \in \iota(X)$ we find that

$$\int_{g \in GL(N,\mathbb{C})} \frac{g \cdot Z_i(x) \overline{g \cdot Z_j(x)}}{\sum_{m=1}^{N} |g \cdot Z_m(x)|^2} d\sigma$$

is a constant multiple of the identity matrix, hence so is

$$\mathbb{E}[\bar{\mu}_X(g \cdot \iota)] = \int_{x \in \iota(X)} \omega_{FS}^n(x) \int_{g \in GL(N,\mathbb{C})} \frac{g \cdot Z_i(x) \overline{g \cdot Z_j(x)}}{\sum_{m=1}^{N} |g \cdot Z_m(x)|^2} d\sigma,$$

as claimed.

Thank you very much for listening!