Collapsing K3 surf

limit measures on interval, Moduli cptification

Y. Odaka.
(Kyoto)

This is a sequel to

My talk (2018), S. Honda, Y. Oshima 's talks (2019)

Recall our background idea

Moduli "b'dry" of polarized Calabi-Yau varieties (X, L)

ample l.b (or Kähler class)

Understand the Relation!

Asymp. Behaviour of Ricci-flat Kähler / Calabi-Yau / Kähler-Einstein metric

Apply

Canonical Cpt'ification of Moduli of CY.

Apply

(More) general lim of CY metric.

So far, most successful in $\boxed{\text{HyperKähler}}$ case, notably K3.

(E.g. KS conj for K3, AV; new inv for type II degen ...)

(Easiest)

Elliptic curve case:

$$X_t = \mathbb{C}^* / t^{\mathbb{Z}} \quad (\text{for } |t| < 1) \qquad \text{"Tate curve"}$$

$$= \mathbb{C} / 2\pi i \mathbb{Z} + \log t \, \mathbb{Z}$$

Néron model

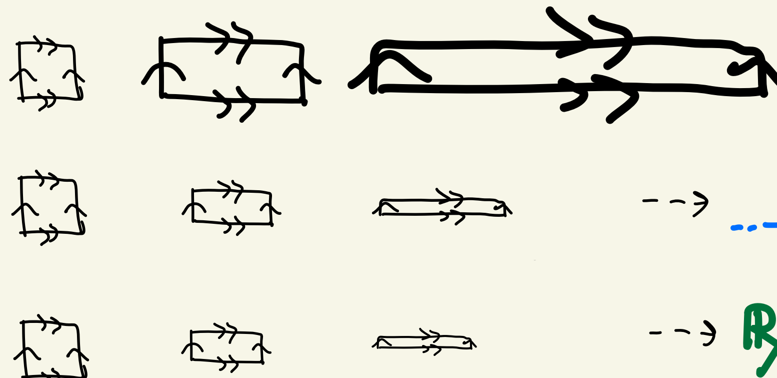
$\Downarrow$

for $t \to 0$

① fix inj.rad (rescale)

② fix vol

③ fix diam (rescale)



$\dashrightarrow \mathbb{C}^*$

$\dashrightarrow \mathbb{R}$

$\dashrightarrow \mathbb{R}/\mathbb{Z} = S^1$

trop. ell curve $\Uparrow$

$\uparrow$

our work.

Brief review of O-Oshima

(talk @ Kanazawa
O'18
Osh'19)

Main conjecture (OO'16)

Explicit determination of Gromov-Hausdorff cptif of Moduli of all K3

(OO'18: S^2 case completed + ⋯)

$\Rightarrow$ Cor (Sun-Zhang '21)

Possible collapsed limits w/ fin diam $\left\{ \begin{array}{l} T^3/\pm 1 \\ S^2 \\ \bullet\!\!-\!\!\bullet \end{array} \right.$

$T^3/\pm 1$, S^2: w/ Hausdorff measure (also cf. Honda san's talk)

$\bullet\!\!-\!\!\bullet$ ⋯ NON triv measure !

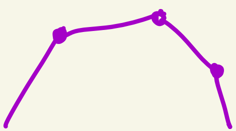
2 years ago, Shouhei Honda san's talks:

Thm (Honda – Sun – Zhang)

If a seq of K3 (X_i, g_i) $i=1,2,\dots$

measured Gromov-Hausdorff converge to interval $[0,1]$,

Limit Measure is of the form



$V\,dx$ (V: PL fun) up to cst

(limit distance $d = \int \sqrt{V}\, dx$

canonical aff. str. $\nabla = dx$.) "Mirror?" $\neq$ Berkovich type aff. str.

... use Cheeger–Tian (finite pts w/ diverging curv, 4-dim)

Cheeger–Fukaya–Gromov (3-dim nilpotent str) $\Rightarrow \nabla$

An observation: Collapsed lim has Hausdorff measure $\Longleftrightarrow$ G = nilpo group (in CFG) is **abelian**

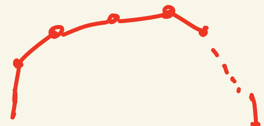
$\underline{Q}$ general?

$\underline{\text{e.g.}}$ HSZ case $\overset{1\text{-dim}}{}$ --- G is Heisenberg group.

$$\boxed{\underline{\text{Question}}:\ \textbf{Explicit}\ \text{understanding of } V}$$

Today:

§1 show explicit classification / determination of V.



§2 via Moduli cptif. again

$\boxed{\text{Approach 1}}$: " tropical + AG " (0)

$\boxed{\text{Approach 2}}$: estimates periods. (Oshima)

§1 List of possible Vs

More precisely, (real) 17-dim Moduli $\mathcal{M}$ of $\{V\}$ explicitly

Step 1 we construct

(Alexeev-Brunyate-Engel, Oshima)

at most 18 non-smooth pts. → V

-- "shape" classified by Dynkin diagram.

Step 2 partially show "these Vs are indeed the [HSZ] lim V"

(O'20, Oshima)

The 17-dim moduli $\mathcal{M}$ of V

$:=$ the real (17)-dim ball quot

$$O(\Lambda_{seg}) \backslash \left\{ v \in \Lambda_{seg} \otimes \mathbb{R} \;\middle|\; v^2 > 0 \right\} / \mathbb{R}_{>0}$$

(isot. plane $P \subset \Lambda_{K3}$ $(3,19)$

$\Lambda_{seg} := P^{\perp}/P$
$= \mathrm{II}_{1,17}$)

$(=: \mathcal{M}_{K3}(a)^{\tau}$ in op. cit)

and $\underbrace{O(\Lambda_{seg}) \backslash \left\{ v \in \Lambda_{seg} \otimes \mathbb{R} \;\middle|\; v^2 > 0 \right\} / \mathbb{R}_{>0}}_{\text{real (17)-dim ball quot}} \xrightarrow{\tilde{\Phi}} \{V\}$

w/ ("$\mathrm{Sym}^2 3$") $\exists$ 6 chambers $\leftrightarrow$ type of V

def by Alexeev-Brunyate-Engel '20 (AG) & Oshima (this DG context) indep!

$\downarrow$

My method follows this, which we explain :

$\boxed{\text{Alg Geom page}}$

Alexeev-Brunyate-Engel '20 :

SLAG fibration!

hK rotate.

$$\mathcal{M}_W = \left\{ \text{(Weierstrass) elliptic K3 surf } \begin{matrix} X \\ \downarrow \\ \mathbb{P}^1 \end{matrix} \right\}$$

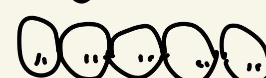
(known to be 18-dim$/\mathbb{C}$ Shimura variety)

$$\overset{\text{open}}{\subset} \ \underset{\text{proj. var.}}{\overline{\mathcal{M}}_W^{ABE}} \quad \text{s.t.} \quad \partial \overline{\mathcal{M}}^{ABE} = \left\{ \underset{\text{variety}}{\overset{\text{(explicit)}}{\text{Singular 2-dim}}} \right\}$$

explicit toroidal cptf.

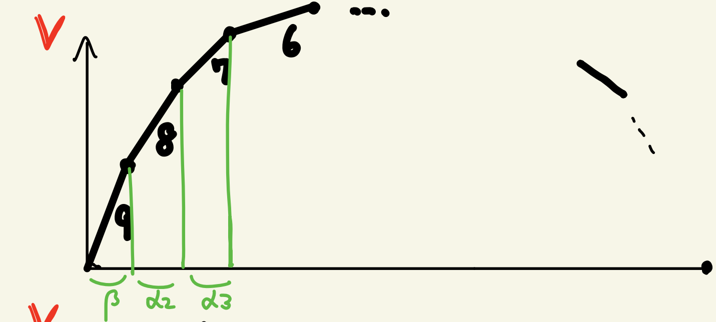
$\downarrow$

chain of $\mathbb{P}^1$

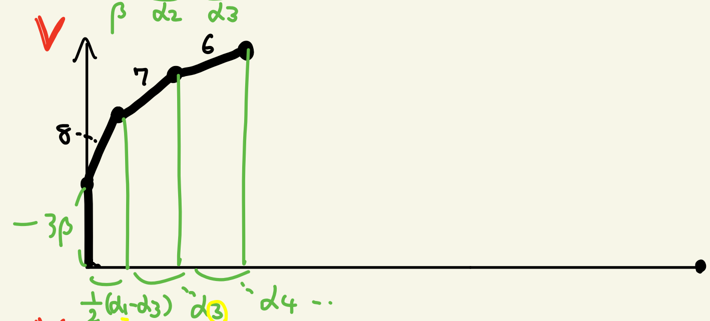


part (i)$_L$

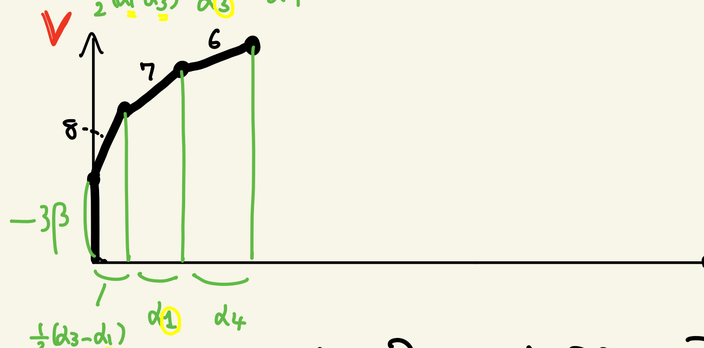
$(\beta, \lambda) \geq 0$



(ii)$_L$



(iii)$_L$



Rmk

[ABE] used

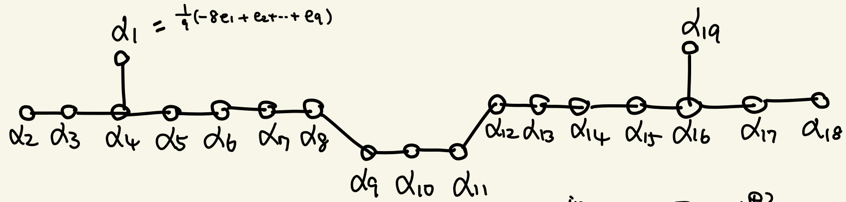
餃子 "dumpling"

for Ellip K3 degen.

the same for Right side.

$\Rightarrow$ 3 (for left) $\times$ 3 (right) patterns $\bmod \mathfrak{S}_2$

.. controlled by the roots



$\alpha_9 = e_8 - e_9 + e$, $\alpha_{10} = -e + f$, $\alpha_{11} = e_8' - e_9' + e$

in $\Lambda_{seg} = E_8(-1)^{\oplus 2} \oplus U$

$E_8(-1)^{\oplus 2} \subset \langle e_1, \dots e_9 \rangle$, $\langle {}', \dots {}' \rangle$; $U = \langle e, f \rangle$

&

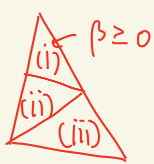
$3\beta := \alpha_1 - 2\alpha_2 - \alpha_3.$

real (17)-dim ball quot

- we identify $O(\Lambda_{seg}) \backslash \{ v \in \Lambda_{seg} \otimes \mathbb{R} \mid v^2 > 0 \} / \mathbb{R}_{>0}$ = Left-Right invol $\backslash \{ \lambda \in \Lambda_{seg, \mathbb{R}} \mid (\alpha_i, \lambda) \geqq 0 \ (\forall i),\ \lambda^2 > 0 \}$

(by Vinberg)

- divide into $(\mathrm{Sym}^2 3 =) 6$ chambers.



$\begin{Bmatrix} (i)_L \\ (ii)_L \\ (iii)_L \end{Bmatrix} \times \begin{Bmatrix} (i)_R \\ (ii)_R \\ (iii)_R \end{Bmatrix}$

(L) decides left end of V
(R) " right "

§2 Determining limit V:

limit measure

$\boxed{\underline{\underline{Q}}\ \text{For } \{X_i\}_{i=1,\cdots},\ \text{how } V \text{ is determined explicitly?}}$

K3 collapsing to $[0,1]$

$\underline{\underline{A}}$ To answer (in terms of periods)

we put the 17-dim moduli $\mathcal{M}$ (of V)

as a boundary strata

of Explicit compactification of

$\boxed{\text{Moduli of K3}}$ $\mathcal{M}_{K3}$

We consider

general Kähler setting

$$\mathcal{F}_{2d} \;(38) \longrightarrow \mathcal{M}_{K3} \;(57) := \left\{ \begin{array}{l} \text{all } \left(\begin{array}{l}\text{KE-metrized}\\ \text{possibly ADE}\end{array}\right) \\ \text{Kähler K3} \end{array} \right\} \Big/ \begin{array}{l}\text{certain}\\ \text{change of } \mathbb{C}\text{-str}\\ (\text{hK rot})\end{array}$$

$\mathcal{F}_{2d} = \{\text{proj K3}\}$

$(\bigcirc : \mathbb{R}\text{-dim})$

$$\mathcal{M}_{K3} \cong SO^{\circ}(3,19) / SO(3) \times SO(19)$$

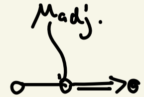
(Kobayashi-Todorov)

cf. 8018

$$\overline{\mathcal{M}_{K3}}^{\text{Satake, adj. rep}} \twoheadleftarrow \overline{\mathcal{M}_{K3}}^{\text{Satake, } \tau} \cdots \text{today}$$

(2 compactifications)

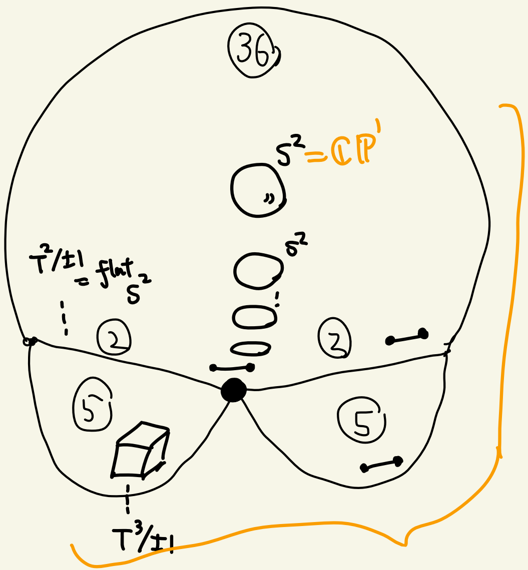
picture



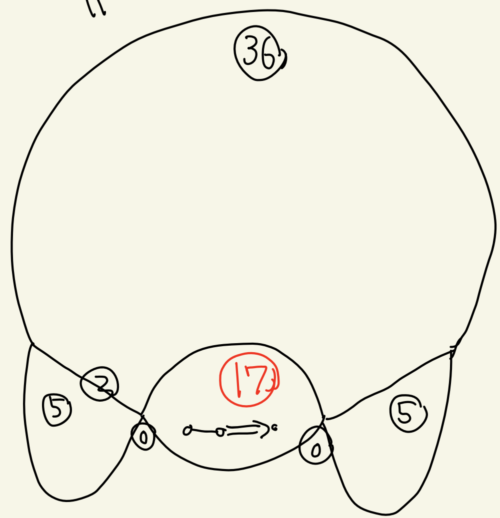
$$\partial \overline{\mathcal{M}_{K3}}^{\text{Satake, adj. rep}} \twoheadleftarrow \partial \overline{\mathcal{M}_{K3}}^{\text{Satake}, \tau}$$

(b'dry) (b'dry)

|| ||



$\twoheadleftarrow$

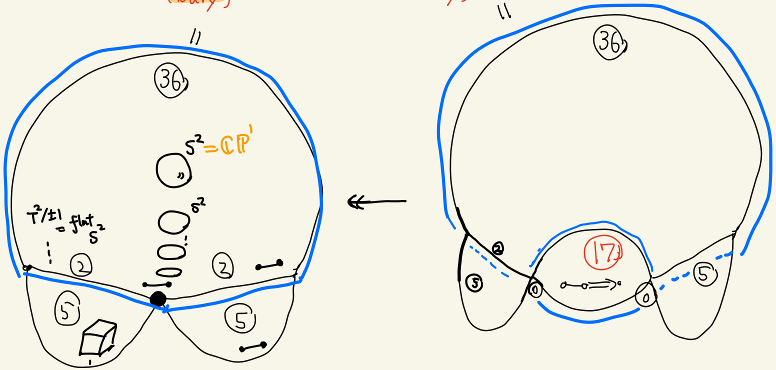


[OO'18 §6] conjectured this is GH cptif.

Extract Blue parts

$\partial \overline{\mathcal{M}}_{K3}^{\text{Satake, adj. rep}}$ (b'dry) $\Longleftarrow$ $\partial \overline{\mathcal{M}}_{K3}^{\text{Satake}, \tau}$ (b'dry)

$\parallel$ $\parallel$



identify $\parallel$

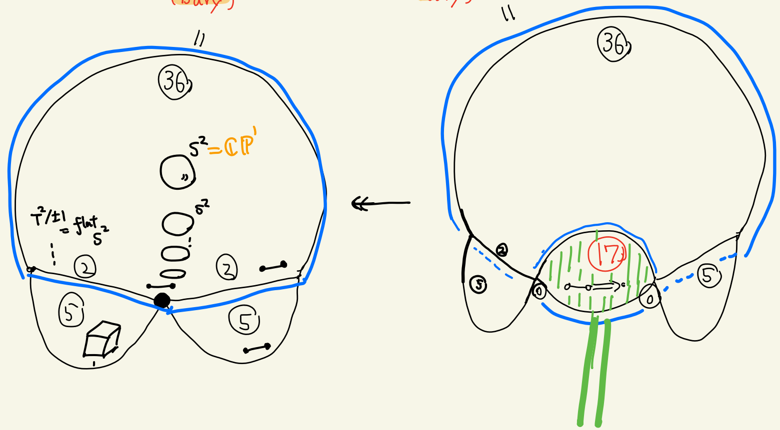
(Weierstrass)
Ellip. K3's Moduli
M_W

18-dim normal **Shimura variety** & its Satake-**Baily-Borel** compactification!

(**GIT** $\parallel$ 0018 §7)

$\partial \overline{\mathcal{M}_{K3}}^{\text{Satake, adj. rep}} \twoheadleftarrow \partial \overline{\mathcal{M}_{K3}}^{\text{Satake}, \tau}$

(bdry) (bdry)



identify

(Weierstrass) ~~Ellip. K3~~'s Moduli

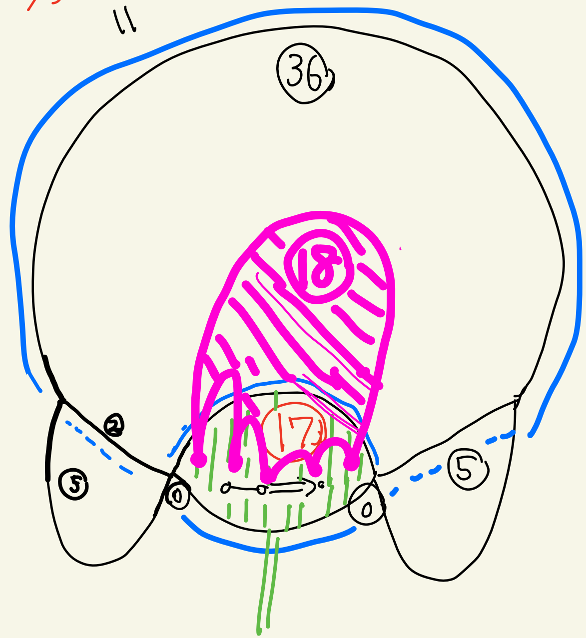
$\{$underlying $\mathbb{P}^1 (\simeq S^2)\}$'s

$\boxed{\text{Nothing but our concerned parameter sp of } V}$

lim of proj K3s

$\partial \overline{\mathcal{M}_{K3}}^{\text{Satake}, \tau}$ (bdry)

=



$\mathcal{M}_{K3}(d)^{\tau}$

Limit locus of $\{\text{proj. K3}\}(F_{2d})$

=

for each d.

$\#(\text{Its} \cap \text{w/ } 17) < \infty$

"but become dense when $\bigcup_{d}$ "

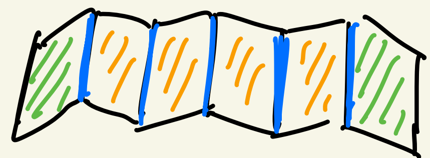
<u>Review</u>

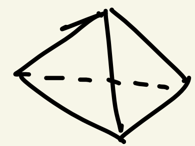
Alg/holom.

(good) degenerations of K3 surf

(Kulikov)

$$\mathcal{X} = \bigcup X_t$$
$$\downarrow$$
$$0 \in \Delta_t \subset \mathbb{C}$$

- type I X_0 : K3 (possibly orbifold)
- type II $X_0 =$ 

  Comb. type : 
- type III $X_0 =$ 

  Comb. type : S^2

$\boxed{\text{Associated lattice}}$ ("weaker info than V")

Λ_{per} for type II degen $(\mathcal{X}^*, \mathcal{L}^*) \subset (\mathcal{X}, \mathcal{L})$

$$\begin{array}{ccc} (\mathcal{X}^*, \mathcal{L}^*) & \subset & (\mathcal{X}, \mathcal{L}) \\ \downarrow & & \downarrow \\ \Delta^* & \subset & \Delta \end{array}$$

$\rightsquigarrow$ 

$$:= \left(v_{2d}^{\perp} \subset P^{\perp}/P \simeq \Lambda_{seg} \right)$$

$[v_{2d} \in P^{\perp}/P]$

$\cup$

DAA⋯AD

or

DA⋯AE

or

EA⋯AE

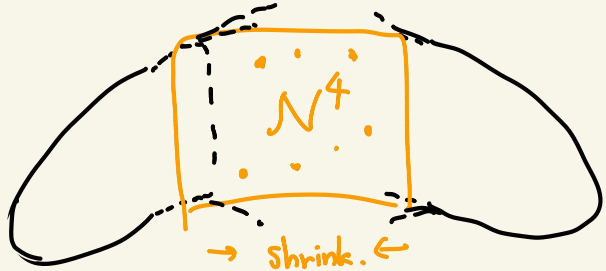
Speculated Geom. meaning (7.1 of op cit [O' arXiv 201000416])

If $V_0 \cup V_1 = \mathcal{X}_0$, then term. obj ('s dim) of MMP w/ scaling $\mathcal{L}|V_i$ on V_i may decide D or E ?

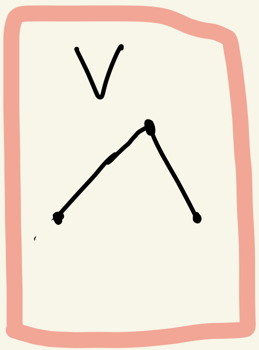
(OK for $d=1$: Friedman 80s)

Example Recall Hein – Sun – Viaclovsky – Zhang '18 :

glued



: Collapsing HK metric on K3



Del Pezzo w/ Tian-Yau metric $\quad \downarrow \pi \quad$ Del Pezzo w/ Tian-Yau metric (vol. growth dim $= \frac{4}{3}$!)

$\pi^{-1}(x) = \Gamma \backslash \text{Heisenberg grp} = \boxed{\begin{matrix} S^1\text{-bdle} \\ \text{over } T^2 \end{matrix}}$ w/ degree jumps

$\Rightarrow$ this is a particular example of type EAE

⊚ Max degen of elliptic K3 $\left[\begin{array}{c} X \\ f \downarrow |e_0| \\ \mathbb{P}^1 \end{array}\right]$

$\rightsquigarrow$ $\exists$ special Lag fibration $X \xrightarrow[|e_1|]{g} B$

OO'18

$\left[\begin{array}{c} \Lambda_{K3} \\ \| \\ \langle e_0, f_0 \rangle \\ \oplus \\ \langle e_1, f_1 \rangle \\ \oplus \\ \mathrm{II}_{1,17} \end{array}\right]$

$\therefore$ 2 Lag fibrations f & g.

[Conj (O'20 (6.2))

HSVZ map π factors through both f & g,

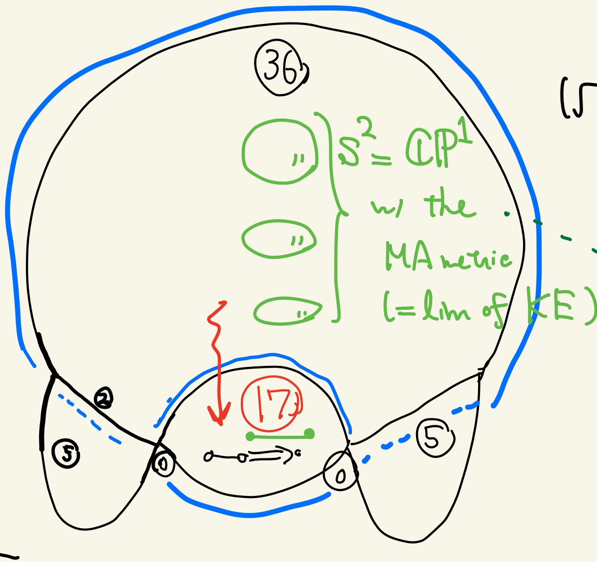
$e_0 \cap e_1$ ($\approx S^1$) is Monodromy invariant (van. cycle) w.r.t. both f & g]

⊚ Another phenomenon!

[Del Pezzo surf $\xrightarrow{\text{LG model}}$ Rational Ell. surf.] is compatible w/ HK rotation (cf Collins-Jacob-Lin)

Main Thm for now ... (O & Oshima)

arXiv 2010.00416 Part II (in preparation)



(Γ of) The explicit V function

$\|$

the $\boxed{\text{limit measure}}$ (on •—•)

of seq of $(S^2, \overset{\mathbb{R}}{\text{MA}} \text{ met})$ "tropical K3"

[Conj: of (K3, KE) as well $\in \mathcal{M}_{K3}$.]

$\mathbb{CP}^1$

$\|$

Recall: S^2 has (generally) 24 singularities $x_1 \sim x_{24}$ discriminants.

$V(g_8^3 - 27 g_{12}^2 = \Delta_{24})$

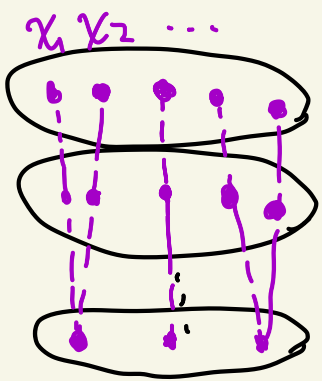
proof of Main Thm

$\boxed{\text{Step 1}}$ • Asym. behaviour of MA metrics on S^2 / $\mathbb{R}$ (cf. [OO18, §7])
estimates via elliptic integral

imply
$\Rightarrow$ Key Limit behaviour of 24 pts $\{x_i\}$ $V(\Delta_{24}) \subset \mathbb{CP}^1$

determines the limit measure $V\,dx$.

especially "when x_i & x_j coincides at lim"



classify all the possibility ... $\circledast$

$\boxed{\text{Step 2}}$ (Main) Classify the possibility of behaviour of $\{x_i\}_{i=1\cdots 24}$

… 2 totally different methods.

via periods

Method 1 (Oshima) : Regard 17-dim (open) bdry strata as boundary strata of Satake cptification as above

· Construct 2 cycles on all K3s reflecting $\{x_i\}$.

· Asym calculation of periods

via tropical

Method 2

(O.
arXiv:2011..)

Regard (17)-dim (open) as b'dry strata

boundary strata of "Morgan-Shalen compactification (= Satake)

key1

||

Dual complex of $\partial \overline{M_W}^{ABE,\nu}$ (toroidal compactification)

& then,

USE Algebraic Geometry (!) of Degenerations/Moduli cptif

(ABE, O Part I)

key2

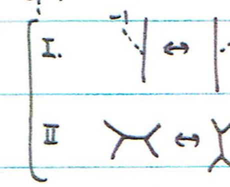
to understand $\boxed{\text{Asym. behaviour of Discriminants}}$

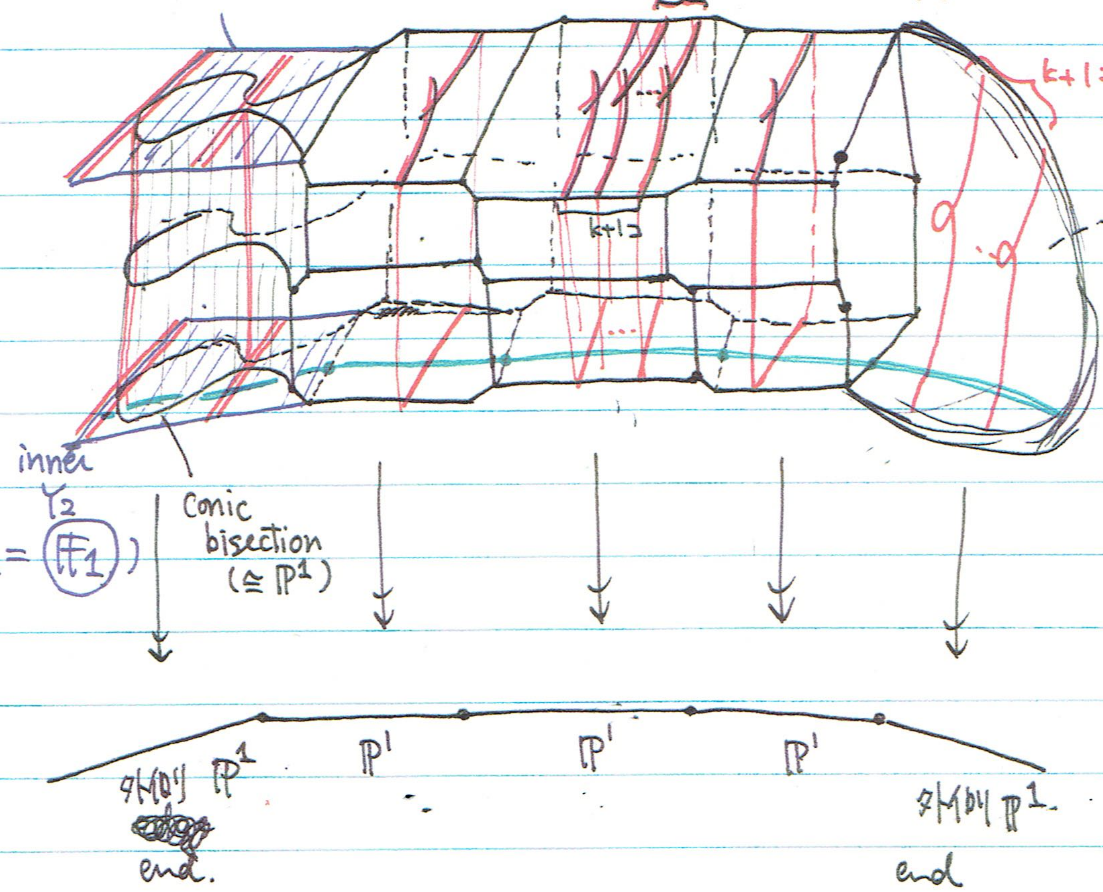
(↳ 24 sing pts in S^2)

PICTURE OF DEGEN. SURF.

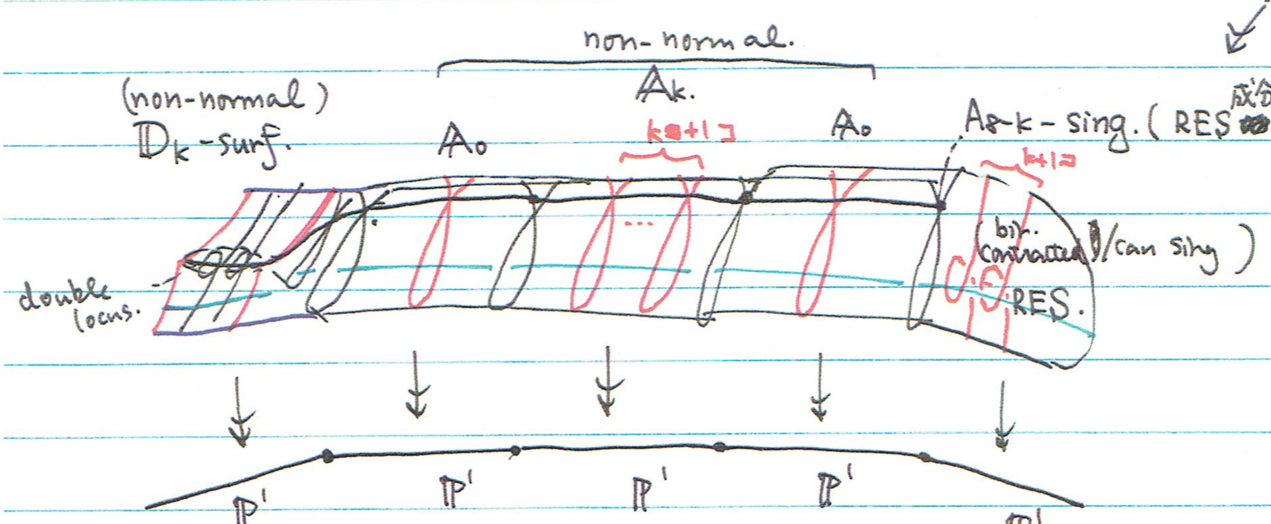
(NOT dual) intersection complex. of $\boxed{\text{Kulikov model}}$

NOT unique.
i.e. allow flops.

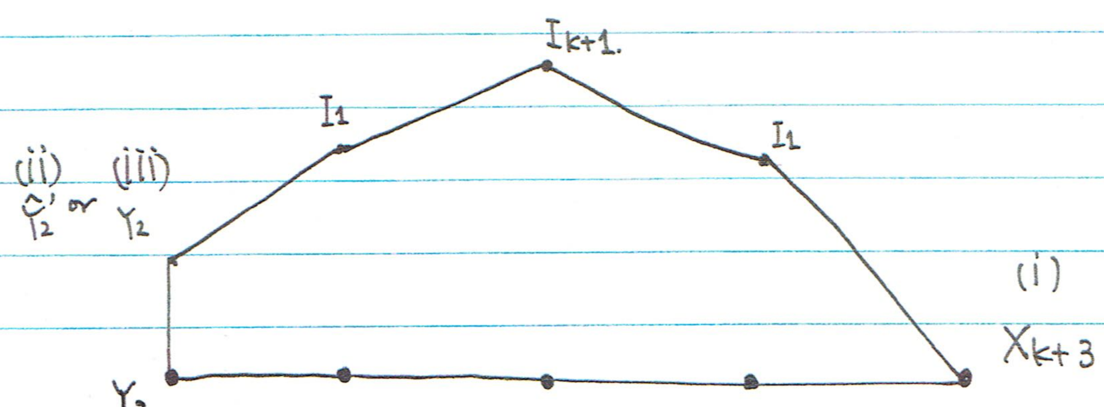




この底面部分のみ残して
他をcontract
(log KSBA)
w/ b'dry = 緑 (section) + …
(× ε, ε << 1)



log KSBA model (unique)



Thank you for your attention!

又金沢にて御会いしましょう！